General Mathematics Exercises

Unit 1. Differentiation

1A. Graphing

1A-1 By completing the square, use translation and change of scale to sketch

a) $y = x^2 - 2x - 1$ b) $y = 3x^2 + 6x + 2$

1A-2 Sketch, using translation and change of scale

a) $y = 1 + |x + 2|$ b) $y = \dfrac{2}{(x-1)^2}$

1A-3 Identify each of the following as even, odd , or neither

a) $\dfrac{x^3 + 3x}{1 - x^4}$ b) $\sin^2 x$

c) $\dfrac{\tan x}{1 + x^2}$ d) $(1 + x)^4$

e) $J_0(x^2)$, where $J_0(x)$ is a function you never heard of

1A-4 a) Show that every polynomial is the sum of an even and an odd function.

b) Generalize part (a) to an arbitrary function $f(x)$ by writing

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) + f(-x)}{2}$$

Verify this equation, and then show that the two functions on the right are respectively even and odd.

c) How would you write $\dfrac{1}{x + a}$ as the sum of an even and an odd function?

1A-5. Find the inverse to each of the following , and sketch both $f(x)$ and the inverse function $g(x)$. Restrict the domain if necessary. (Write $y = f(x)$ and solve for y; then interchange x and y.)

a) $\dfrac{x - 1}{2x + 3}$) b) $x^2 + 2x$

1A-6 Express in the form $A \sin(x + c)$

a) $\sin x + \sqrt{3} \cos x$ b) $\sin x - \cos x$

1A-7 Find the period , amplitude , and phase angle, and use these to sketch

a) $3 \sin(2x - \pi)$ b) $-4 \cos(x + \pi/2)$

1A-8 Suppose $f(x)$ is odd and periodic. Show that the graph of $f(x)$ crosses the x-axis infinitely often.

1A-9 a) Graph the function f that consist of straight line segments joining the points $(-1,-1)$, $(1,2)$, $(3,-1)$, and $(5,2)$. Such a function is called piecewise linear.

b) Extend the graph of f periodically. What is its period?

c) Graph the function $g(x) = 3f((x/2) - 1) - 3$.

1B. Velocity and rates of change

1B-1 A test tube is knocked off a tower at the top of the Green building. (For the purposes of this experiment the tower is 400 feet above the ground, and all the air in the vicinity of the Green building was evacuated, so as to eliminate wind resistance.) The test tube drops $16t^2$ feet in t seconds. Calculate

a) the average speed in the first two seconds of the fall

b) the average speed in the last two seconds of the fall

c) the instantaneous speed at landing

1B-2 A tennis ball bounces so that its initial speed straight upwards is b feet per second. Its height s in feet at time t seconds is given by $s = bt - 16t^2$

a) Find the velocity $v = ds/dt$ at time t.

b) Find the time at which the height of the ball is at its maximum height.

c) Find the maximum height.

d) Make a graph of v and directly below it a graph of s as a function of time. Be sure to mark the maximum of s and the beginning and end of the bounce.

e) Suppose that when the ball bounces a second time it rises to half the height of the first bounce. Make a graph of s and of v of both bounces, labelling the important points. (You will have to decide how long the second bounce lasts and the initial velocity at the start of the bounce.)

f) If the ball continues to bounce, how long does it take before it stops?

1C. Slope and derivative

1C-1 a) Use the difference quotient definition of derivative to calculate the rate of change of the area of a disk with respect to its radius. (Your answer should be the circumference of the disk.)

b) Use the difference quotient definition of derivative to calculate the rate of change of the volume of a ball with respect to the radius. (Your answer should be the surface area of the ball.)

1C-2 Let $f(x) = (x-a)g(x)$. Use the definition of the derivative to calculate that $f'(a) = g(a)$, assuming that g is continuous.

1C-3 Calculate the derivative of each of these functions directly from the definition.

a) $f(x) = 1/(2x+1)$ b) $f(x) = 2x^2 + 5x + 4$

c) $f(x) = 1/(x^2+1)$ d) $f(x) = 1/\sqrt{x}$

e) For part (a) and (b) find points where the slope is $+1$, -1, 0.

1C-4 Write an equation for the tangent line for the following functions

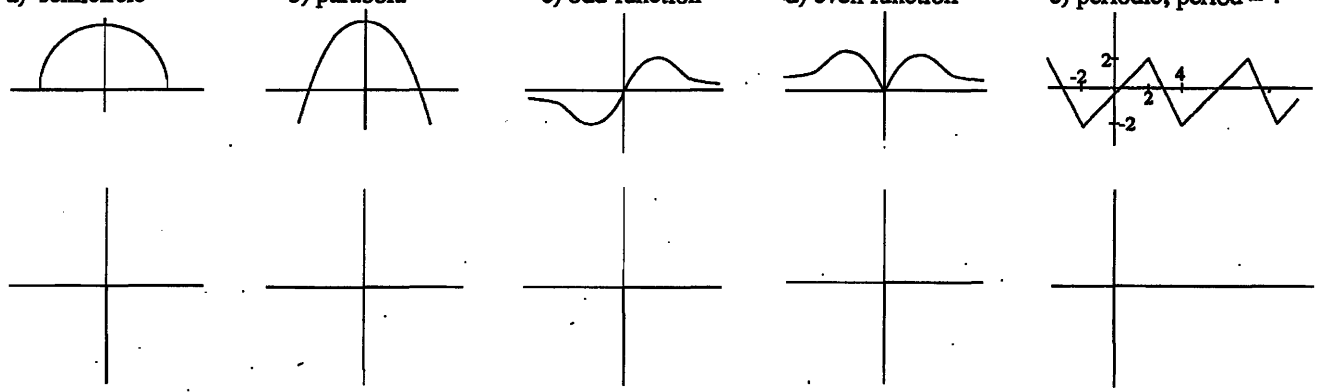
a) $f(x) = 1/(2x+1)$ at $x = 1$ b) $f(x) = 2x^2 + 5x + 4$ at $x = a$

c) $f(x) = 1/(x^2+1)$ at $x = 0$ d) $f(x) = 1/\sqrt{x}$ at $x = a$

1C-5 Find all tangent lines through the origin to the graph of $y = 1 + (x-1)^2$.

1C-6 Graph the derivative of the following functions directly below the graph of the function. It is very helpful to know that the derivative of an odd function is even and the derivative of an even function is odd (see **1F-6**).

a) semicircle b) parabola c) odd function d) even function e) periodic; period = ?



1D. Limits and continuity

1D-1 Calculate the following limits if they exist. If they do not exist, then indicate whether they are $+\infty$, $-\infty$ or undefined.

a) $\displaystyle\lim_{x\to 0} \frac{4}{x-1}$ b) $\displaystyle\lim_{x\to 2} \frac{4x}{x+1}$ c) $\displaystyle\lim_{x\to -2} \frac{4x^2}{x+2}$

d) $\displaystyle\lim_{x\to 2+} \frac{4x^2}{2-x}$ e) $\displaystyle\lim_{x\to 2-} \frac{4x^2}{2-x}$ f) $\displaystyle\lim_{x\to \infty} \frac{4x^2}{x-2}$

g) $\displaystyle\lim_{x\to \infty} \frac{4x^2}{x-2} - 4x$ i) $\displaystyle\lim_{x\to \infty} \frac{x^2+2x+3}{3x^2-2x+4}$ j) $\displaystyle\lim_{x\to 2} \frac{x-2}{x^2-4}$

1D-2 For which of the following should one use the one-sided limit? Evaluate it.

a) $\displaystyle\lim_{x\to 0} \sqrt{x}$ b) $\displaystyle\lim_{x\to 1} \frac{1}{x-1}$ c) $\displaystyle\lim_{x\to 1} \frac{1}{(x-1)^4}$ d) $\displaystyle\lim_{x\to 0} |\sin x|$ e) $\displaystyle\lim_{x\to 0} \frac{|x|}{x}$

1D-3 Identify and give the type of the points of discontinuity of each of the following:

a) $\dfrac{x-2}{x^2-4}$ b) $\dfrac{1}{\sin x}$ c) $\dfrac{x^4}{x^3}$ d) $f(x) = \begin{cases} x+a, & x>0 \\ a-x, & x<0 \end{cases}$

e) $f'(x)$, for the $f(x)$ in d) f) $(f(x))^2$, where $f(x) = \dfrac{d}{dx}|x|$

1D-4 Graph the following functions.

a) $\dfrac{4x^2}{x-2}$ (See **1D-1efg.**) b) $\dfrac{1}{x^2+2x+2}$

1D-5 Define $f(x) = \begin{cases} ax+b, & x \ge 1; \\ x^2, & x < 1. \end{cases}$

a) Find all values of a, b such that $f(x)$ is continuous.

b) Find all values of a, b such that $f'(x)$ is continuous. (Be careful!)

1D-6 For each of the following functions, find all values of the constants a and b for which the function is differentiable.

a) $f(x) = \begin{cases} x^2+4x+1, & x \ge 0; \\ ax+b, & x < 0. \end{cases}$ b) $f(x) = \begin{cases} x^2+4x+1, & x \ge 1; \\ ax+b, & x < 1. \end{cases}$

1D-7 Find the values of the constants a, b and c for which the following function is differentiable. (Give a and b in terms of c.)

$$f(x) = \begin{cases} cx^2+4x+1, & x \ge 1; \\ ax+b, & x < 1. \end{cases}$$

1D-8 For each of the following functions, find the values of the constants a and b for which the function is continuous, but *not* differentiable.

a) $f(x) = \begin{cases} ax+b, & x > 0; \\ \sin 2x, & x \le 0. \end{cases}$ b) $f(x) = \begin{cases} ax+b, & x > 0; \\ \cos 2x, & x \le 0. \end{cases}$

1D-9 Find the values of the constants a and b for which the following function is differentiable, but *not* continuous.

$$f(x) = \begin{cases} ax+b, & x > 0; \\ \cos 2x, & x \le 0. \end{cases}$$

1D-10* Show that

$$g(h) = \frac{f(a+h)-f(a)}{h} \quad \text{has a removable discontinuity at } h=0 \iff f'(a) \text{ exists.}$$

1E. Differentiation formulas: polynomials, products, quotients

1E-1 Find the derivative of the following polynomials.

a) $x^{10}+3x^5+2x^3+4$ b) e^2+1 ($e =$ base of natural logs)

c) $x/2+\pi^3$ d) $(x^3+x)(x^5+x^2)$

1E-2 Find the antiderivative of the following polynomials.

a) $ax+b$ (a and b are constants.) b) $x^6+5x^5+4x^3$

c) $(x^3+1)^2$

1E-3 Find the points (x, y) of the graph $y = x^3+x^2-x+2$ at which the slope of the tangent line is horizontal.

1E-4 For each of the following, find all values of a and b for which $f(x)$ is differentiable.

a) $f(x) = \begin{cases} ax^2+bx+4, & x \le 0; \\ 5x^5+3x^4+7x^2+8x+4, & x > 0. \end{cases}$ b) $f(x) = \begin{cases} ax^2+bx+4, & x \le 1; \\ 5x^5+3x^4+7x^2+8x+4, & x > 1. \end{cases}$

1E-5 Find the derivatives of the following rational functions.

a) $\dfrac{x}{1+x}$ b) $\dfrac{x+a}{x^2+1}$ (a is constant)

c) $\dfrac{x+2}{x^2-1}$ d) $\dfrac{x^4+1}{x}$

1F. Chain rule, implicit differentiation

1F-1 Find the derivative of the following functions:

a) $(x^2+2)^2$ (two methods)

b) $(x^2+2)^{100}$. Which of the two methods from part (a) do you prefer?

1F-2 Find the derivative of $x^{10}(x^2+1)^{10}$.

1F-3 Find dy/dx for $y = x^{1/n}$ by implicit differentiation.

1F-4 Calculate dy/dx for $x^{1/3} + y^{1/3} = 1$ by implicit differentiation. Then solve for y and calculate y' using the chain rule. Confirm that your two answers are the same.

1F-5 Find all points of the curve(s) $\sin x + \sin y = 1/2$ with horizontal tangent lines. (This is a collection of curves with a periodic, repeated pattern because the equation is unchanged under the transformations $y \to y + 2\pi$ and $x \to x + 2\pi$.)

1F-6 Show that the derivative of an even function is odd and that the derivative of an odd function is even.

(Write the equation that says f is even, and differentiate both sides, using the chain rule.)

1F-7 Evaluate the derivatives. Assume all letters represent constants, except for the independent and dependent variables occurring in the derivative.

a) $D = \sqrt{(x-a)^2 + y_0{}^2}$, $\dfrac{dD}{dx} = ?$ b) $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$, $\dfrac{dm}{dv} = ?$

c) $F = \dfrac{mg}{(1+r^2)^{3/2}}$, $\dfrac{dF}{dr} = ?$ d) $Q = \dfrac{at}{(1+bt^2)^3}$, $\dfrac{dQ}{dt} = ?$

1F-8 Evaluate the derivative by implicit differentiation. (Same assumptions about the letters as in the preceding exercise.)

a) $V = \dfrac{1}{3}\pi r^2 h$, $\dfrac{dr}{dh} = ?$ b) $PV^c = nRT$, $\dfrac{dP}{dV} = ?$

c) $c^2 = a^2 + b^2 - 2ab\cos\theta$, $\dfrac{da}{db} = ?$

1G. Higher derivatives

1G-1 Calculate y'' for the following functions.

a) $3x^2 + 2x + 4\sqrt{x}$ b) $\dfrac{x}{x+5}$

c) $\dfrac{-5}{x+5}$ d) $\dfrac{x^2+5x}{x+5}$

1G-2 Find all functions $f(x)$ whose third derivative $f'''(x)$ is identically zero. ("Identically" is math jargon for "always" or "for every value of x".)

1G-3 Calculate y'' using implicit differentiation and simplify as much as possible.

$$x^2a^2 + y^2b^2 = 1$$

1G-4 Find the formula for the nth derivative $y^{(n)}$ of $y = 1/(x+1)$.

1G-5 Let $y = u(x)v(x)$.

a) Find y', y'', and y'''.

b) The general formula for $y^{(n)}$, the n-th derivative, is called **Leibniz' formula**: it uses the same coefficients as the binomial theorem , and looks like

$$y^{(n)} = u^{(n)}v + \binom{n}{1}u^{(n-1)}v^{(1)} + \binom{n}{2}u^{(n-2)}v^{(2)} + \ldots + uv^{(n)}$$

Use this to check your answers in part (a), and use it to calculate $y^{(p+q)}$, if $y = x^p(1+x)^q$.

1H. Exponentials and Logarithms: Algebra

1H-1 The *half-life* λ of a radioactive substance dacaying according to the law $y = y_0e^{kt}$ is defined to be the time it takes the amount to decrease to 1/2 of the initial amount y_0.

a) Express the half-life λ in terms of k. (Do this from scratch — don't just plug into formulas given here or elsewhere.)

b) Show using your expression for λ that if at time t_1 the amount is y_1, then at time $t_1 + \lambda$ it will be $y_1/2$, no matter what t_1 is.

1H-2 If a solution containing a heavy concentration of hydrogen ions (i.e., a strong acid) is diluted with an equal volume of water, by approximately how much is its pH changed? (Express $(\text{pH})_{\text{diluted}}$ in terms of $(\text{pH})_{\text{original}}$.)

1H-3 Solve the following for y:

a) $\ln(y+1) + \ln(y-1) = 2x + \ln x$ b) $\log(y+1) = x^2 + \log(y-1)$

c) $2\ln y = \ln(y+1) + x$

1H-4 Solve $\dfrac{\ln a}{\ln b} = c$ for a in terms of b and c; then repeat, replacing ln by log.

1H-5 Solve for x (hint: put $u = e^x$, solve first for u):

a) $\dfrac{e^x + e^{-x}}{e^x - e^{-x}} = y$ b) $y = e^x + e^{-x}$

1H-6 Evaluate from scratch the number $A = \log e \cdot \ln 10$. Then generalize the problem, and repeat the evaluation.

1H-7 The decibel scale of loudness is

$$L = 10\log_{10}(I/I_0)$$

where I, measured in watts per square meter, is the intensity of the sound and $I_0 = 10^{-12}$ watt/m^2 is the softest audible sound at 1000 hertz. Classical music typically ranges from 30 to 100 decibels. The human ear's pain threshold is about 120 decibels.

a) Suppose that a jet engine at 50 meters has a decibel level of 130, and a normal conversation at 1 meter has a decibel level of 60. What is the ratio of the intensities of the two sounds?

b) Suppose that the intensity of sound is proportional to the inverse square of the distance from the sound. Based on this rule, calculate the decibel level of the sound from the jet at a distance of 100 meters, at distance of 1 km.[1]

1H-8* The mean distance of each of the planets to the Sun and their mean period of revolution is as follows.[2] (Distance is measured in millions of kilometers and time in Earth years.)

Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto
57.9	108	150	228	778	1,430	2,870	4,500	5,900
0.241	0.615	1.00	1.88	11.9	29.5	84.0	165	248

a) Find the pattern in these data by making a graph of $(\ln x, \ln y)$ where x is the distance to the Sun and y is the period of revolution for the first four points (Mercury through Mars). Observe that these points are nearly on a straight line. Plot a line with ruler and estimate its slope. (You can check your estimated slope by calculating slopes of lines connecting consecutive data points.)

b) Using an approximation to the slope m that you found in part (a) accurate to two significant figures, give a formula for y in the form

$$\ln y = m \ln x + c$$

(Use the Earth to evaluate c.)

c) Solve for y and make a table for the predicted values of the periods of revolutions of all the planets based on their distance to the Sun. (Your answers should be accurate to one percent.)

d) The Earth has radius approximately $6,000$ km and the Moon is at a distance of about $382,000$ km. The period of revolution of the Moon is a lunar month, say 28 days. Assume that the slope m is the same for revolution around the Earth as the on you found for revolution around the Sun in (a). Find the distance above the surface of the Earth of geosynchronous orbit, that is, the altitude of the orbit of a satellite that stays above one place on the equator. (For satellites this close to Earth it is important to know that y is predicting the distance from the satellite to the center of the Earth. This is why you need to know the radius of the Earth.)

e) Find the period of revolution of a satellite that circles at an altitude of $1,000$ km.

[1] The inverse square law is justified by the fact that the intensity is measured in energy per unit time per unit area. When the sound has travelled a distance r, the energy of a sound spread over a sphere of radius r centered at the source. The area of that sphere is proportional to r^2, so the average intensity is proportional to $1/r^2$. Fortunately for people who live near airports, sound doesn't travel as well as this. Part of the energy is dissipated into heating the air and another part into vibration of insulating materials on the way to the listener's ear.

[2] from "Fundamentals of Physics, vol. 1," by D. Halliday and R. Resnick

1I. Exponential and Logarithms: Calculus

1I-1 Calculate the derivatives

a) xe^x b) $(2x-1)e^{2x}$ c) e^{-x^2}

d) $x\ln x - x$ e) $\ln(x^2)$ f) $(\ln x)^2$

g) $(e^{x^2})^2$ h) x^x i) $(e^x + e^{-x})/2$

j) $(e^x - e^{-x})/2$ k) $\ln(1/x)$ l) $1/\ln x$

m) $(1-e^x)/(1+e^x)$

1I-2 Graph the function $y = (e^x + e^{-x})/2$.

1I-3 a) Evaluate $\lim_{n\to\infty} n\ln(1+\frac{1}{n})$. Hint: Let $h = 1/n$, and use $(d/dx)\ln(1+x)|_{x=0} = 1$.

b) Deduce that $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

1I-4 Using $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$, calculate

a) $\lim_{n\to\infty}(1+\frac{1}{n})^{3n}$ b) $\lim_{n\to\infty}(1+\frac{2}{n})^{5n}$ c) $\lim_{n\to\infty}(1+\frac{1}{2n})^{5n}$

1I-5* If you invest P dollars at the annual interest rate r, then after one year the interest is $I = rP$ dollars, and the total amount is $A = P + I = P(1+r)$. This is *simple interest.*

For *compound interest*, the year is divided into k equal time periods and the interest is calculated and added to the account at the end of each period. So at the end of the first period, $A = P(1+r(\frac{1}{k}))$; this is the new amount for the second period, at the end of which $A = P(1+r(\frac{1}{k}))(1+r(\frac{1}{k}))$, and continuing this way, at the end of the year the amount is

$$A = P\left(1+\frac{r}{k}\right)^k .$$

The compound interest rate r thus earns the same in a year as the simple interest rate of

$$\left(1+\frac{r}{k}\right)^k - 1 ;$$

this equivalent simple interest rate is in bank jargon the "annual percentage rate" or APR.[3]

a) Compute the APR of 5% compounded monthly, daily,[4] and continuously. Continuous compounding means the limit as k tends to infinity.

b) As in part (a), compute the APR of 10% compounded monthly, biweekly (k=26), daily, and continuously. (We have thrown in the biweekly rate because loans can be paid off biweekly.)

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[3]Banks are required to reveal this so-called APR when they offer loans. The APR also takes into account certain bank fees known as points. Unfortunately, not all fees are included in it, and the true costs are higher if the loan is paid off early.

[4]For daily compounding assume that the year has 365 days, not 365.25. Banks are quite careful about these subtle differences. If you look at official tables of rates from precalculator days you will find that they are off by small amounts because U.S. regulations permitted banks to pretend that a year has 360 days.

1J. Trigonometric functions

1J-1 Calculate the derivatives of the following functions

a) $\sin(5x^2)$	b) $\sin^2(3x)$	c) $\ln(\cos(2x))$
d) $\ln(2\cos x)$	e) $\dfrac{\sin x}{x}$	f) $\cos(x+y)$; $y = f(x)$
g) $\cos(x+y)$; y constant	h) $e^{\sin^2 x}$	i) $\ln(x^2 \sin x)$
j) $e^{2x}\sin(10x)$	k) $\tan^2(3x)$	l) $\sec\sqrt{1-x^2}$

m) The following three functions have the same derivative: $\cos(2x)$, $\cos^2 x - \sin^2 x$, and $2\cos^2 x$. Verify this. Are the three functions equal? Explain.

n) $\sec(5x)\tan(5x)$	o) $\sec^2(3x) - \tan^2(3x)$	p) $\sin(\sqrt{x^2+1})$
q) $\cos^2(\sqrt{1-x^2})$	r) $\tan^2(\dfrac{x}{x+1})$	

1J-2 Calculate $\displaystyle\lim_{x\to\pi/2} \frac{\cos x}{x - \pi/2}$ by relating it to a value of $(\cos x)'$.

1J-3 a) Let $a > 0$ be a given constant. Find in terms of a the value of $k > 0$ for which $y = \sin(kx)$ and $y = \cos(kx)$ both satisfy the equation

$$y'' + ay = 0.$$

Use this value of k in each of the following parts.

b) Show that $y = c_1\sin(kx) + c_2\cos(kx)$ is also a solution to the equation in (a), for any constants c_1 and c_2.

c) Show that the function $y = \sin(kx + \phi)$ (whose graph is a sine wave with phase shift ϕ) also satisfies the equation in (a), for any constant ϕ.

d) Show that the function in (c) is already included among the functions of part (b), by using the trigonometric addition formula for the sine function. In other words, given k and ϕ, find values of c_1 and c_2 for which

$$\sin(kx + \phi) = c_1\sin(kx) + c_2\cos(kx)$$

1J-4 a) Show that a chord of the unit circle with angle θ has length $\sqrt{2 - 2\cos\theta}$. Deduce from the half-angle formula

$$\sin(\theta/2) = \sqrt{\frac{1-\cos\theta}{2}}$$

that the length of the chord is

$$2\sin(\theta/2)$$

b) Calculate the perimeter of an equilateral n-gon with vertices at a distance 1 from the center. Show that as n tends to infinity, the perimeter tends to 2π, the circumference of the unit circle.

2. Applications of Differentiation

2A. Approximation

2A-1 Find the linearization of $\sqrt{a+bx}$ at 0, by using (2), and also by using the basic approximation formulas. (Here a and b are constants; assume $a > 0$. Do not confuse this a with the one in (2), which has the value 0.)

2A-2 Repeat Exercise A-1 for the function $\dfrac{1}{a+bx}$, $\quad a \neq 0$.

2A-3 Find the linearization at 0 of $\dfrac{(1+x)^{3/2}}{1+2x}$ by using the basic approximation formulas, and also by using (2).

2A-4 Find the linear approximation for $h \approx 0$ for $w = \dfrac{gW}{(1+h/R)^2}$, the weight of a body at altitude h above the earth's surface, where W is the surface weight and R is the radius of the earth. (Do this without referring to the notes.)

2A-5 Making reasonable assumptions, if a person 5 feet tall weighs on the average 120 lbs., approximately how much does a person 5′1″ tall weigh?

2A-6 Find a quadratic approximation to $\tan\theta$, for $\theta \approx 0$.

2A-7 Find a quadratic approximation to $\dfrac{\sec x}{\sqrt{1-x^2}}$, for $x \approx 0$.

2A-8 Find the quadratic approximation to $1/(1-x)$, for $x \approx 1/2$. (Either use (13), or put $x = \frac{1}{2} + h$ and use the basic approximations.)

2A-9* Derive (12) algebraically as suggested in the Notes.

2A-10 Derive (9), (10), (12) by using formula (13).

2A-11 For an ideal gas at constant temperature, the variables p (pressure) and v (volume) are related by the equation $pv^k = C$, where k and C are constants. If the volume is changed slightly from v_0 to $v_0 + \Delta v$, what quadratic approximation expressing p in terms of Δv would you use? (Find the approximation valid for $\Delta v \approx 0$.)

2A-12 Give the indicated type of approximation at the point indicated. (This is to be done after studying the log and exponential functions.)

a) $\dfrac{e^x}{1-x}$ (quadratic, $x \approx 0$) b) $\dfrac{\ln(1+x)}{xe^x}$ (linear, $x \approx 0$)

c) e^{-x^2} (quadratic, $x \approx 0$) d) $\ln\cos x$ (quadratic, $x \approx 0$)

e) $x \ln x$ (quadratic, $x \approx 1$) (Hint: put $x = 1+h$.)

2A-13 Find the linear and quadratic approximation to the following functions

a) $\sin(2x)$, near 0 b) $\cos(2x)$, near 0 c) $\sec(x)$, near 0

d) e^{x^2}, near 0 e) $1/(a+bx)$, near 0; assuming $a \neq 0$;

f) $1/(a+bx)$, near 1; what do you have to assume about the consants a and b?

2A-14* Suppose that a piece of bubble gum has volume 4 cubic centimeters.

a) Use a linear approximation to calculate the thickness of a bubble of inner radius 10 centimeters.

(Start with the relation between the volume V of a sphere and the radius r, and derive the approximate relation between ΔV and Δr.)

b) Find the exact answer.

c) To how many significant figures is the linear approximation accurate? In other words, find the order of magnitude of the difference between the approximation and the exact answer. (Be sure you use enough digits of π to reflect correctly this accuracy!)

d) Use a quadratic approximation to the exact formula for the thickness that you found in part (b) to get an even more accurate estimate.

e) Why is the quadratic term comparable to the error in the accuracy of the linear approximation?

2A-15 Find the linear and quadratic approximations to $\cos(3x)$ near $x = 0$, $\pi/6$, and $\pi/3$.

2A-16 a) Use the law of cosines to find the formula $\sqrt{2 - \cos(2\pi/n)}$ for the length of the side of the equilateral n-gon inscribed in the unit circle.

b) Compute the perimeter and then compute the limit as n tends to infinity using the quadratic approximation to $\cos\theta$ near $\theta = 0$. (Compare with 1J-4.)

2B. Curve Sketching

2B-1 Sketch the graphs of the following. Find the intervals on which it is increasing and decreasing and decide how many solutions there are to $y = 0$. (Graphs need not reflect inflection points, which are discussed in **9-2**).

a) $y = x^3 - 3x + 1$

b) $y = x^4 - 4x + 1$

c) $y'(x) = 1/(1 + x^2)$ and $y(0) = 0$.

d) $y = x^2/(x - 1)$

e) $y = x/(x + 4)$

f) $y = \sqrt{x+1}/(x - 3)$

g) $y = 3x^4 - 16x^3 + 18x^2 + 1$

h) $y = e^{-x^2}$

i) $y' = e^{-x^2}$ and $y(0) = 0$.

2B-2 Find the inflection points of the graphs in problem 1.

2B-3 Find the conditions on a, b and c for which the cubic

$$y = x^3 + ax^2 + bx + c$$

has a local maximum and a local minimum. Use the following two methods:

a) Find the condition under which y' has two distinct real roots. Which of these roots is at the local maximum and which is at the local minimum? (Draw a picture.)

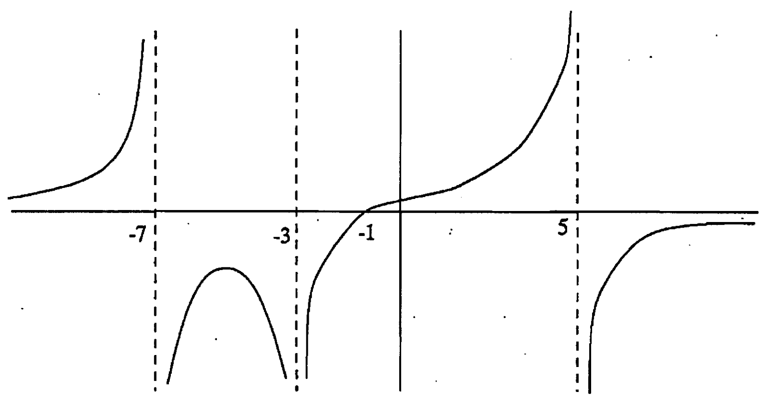
b) Find the condition under which $y' < 0$ at the inflection point. Why does this property imply that there is a local maximum and a local minimum?

2B-4 Suppose that f is a continuous function on $0 \le x \le 10$. Sketch the graph from the following description: f is zero at 4, 7, and 9. $f'(x) > 0$ on $0 < x < 5$ and $8 < x < 10$

and $f'(x) < 0$ on $5 < x < 8$. With the given information, can you say anything for certain about the maximum value, the minimum value of f? Can you say anything about the place where the maximum is attained or the place where the minimum is attained?

2B-5 a) Trace a copy of the graph of the function below and draw the graph of the derivative directly underneath. Connect the inflection point to the corresponding point on the graph of the derivative with a vertical dotted line.

b) Find a rational function with a graph resembling the one below.



2B-6 a) Find a cubic polynomial with a local maximum at $x = -1$ and a local minimum at $x = 1$.

b) Draw the graph of the cubic on $-3 \leq x \leq 3$.

c) Draw a differentiable function on $-3 \leq x \leq 3$ that has an absolute maximum at $x = -1$ and an absolute minimum at $x = 1$.

2B-7 a) Prove that if $f(x)$ is increasing and it has a derivative at a, then $f'(a) \geq 0$. (You may use the fact that a positive function has a limit ≥ 0.)

b) If the conclusion of part (a) is changed to : $f'(a) > 0$, the statement becomes false. Indicate why the proof of part (a) fails to show that $f'(a) > 0$, and give a counterexample to the conclusion $f'(a) > 0$ (i.e., an example for which it is false).

c) Prove that if $f(x)$ has relative maximum at a and it has a derivative at a, then $f'(a) = 0$. (Consider the right-and left-hand limits of $\Delta y/\Delta x$; apply the ideas of part (a).)

2C. Max-min problems

2C-1 Cut four identical squares out of the corners of a 12 by 12 inch piece of cardboard and fold the sides so as to make a box without a top. Find the size of the corner square that maximizes the volume of the box.

2C-2 You are asked to design a rectangular barnyard enclosing 20,000 square feet with fencing on three sides and the wall of a long barn on the fourth. Find the shortest length of fence needed.

2C-3 What is the largest value of the product xy, if x and y are related by $x + 2y = a$, where a is a fixed positive constant?

2C-4 The U. S. Postal Service accepts boxes whose length plus girth equals at most 108 inches. What are the dimensions of the box of largest volume that is accepted? What is its volume (in cubic feet)?

("Length" is the longest of the three dimensions and "girth" is the sum of the lengths of the four sides of the face perpendicular to the length, that is, the "waist-line" measurement of the box. Note that the best shape for this rectangular face perpendicular to the length must be a square.)

2C-5 Find the proportions of the open (i.e., topless) cylindrical can having the largest volume inside, among all those having a fixed surface area A. (Use as the variable the radius r.)

2C-6 What are the largest and the smallest possible values taken on by the product of three distinct numbers spaced so the central number x has distance 1 from each of the other two, if x lies between -2 and 2 inclusive?

2C-7 Find the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius a.

2C-8 Find the dimensions of the rectangle of largest area in a right triangle, if

a) the sides of the rectangle are parallel to the legs;

b) one side of the rectangle is parallel to the hypotenuse.

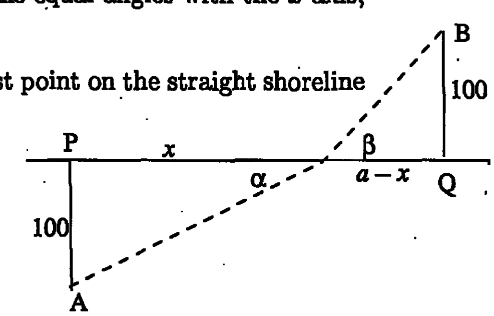
(Of the two placements, which gives the rectangle of larger area?)

2C-9 A light ray reflected in a mirror travels the shortest distance between its starting point and its endpoint.

Suppose the ray starts at $(0,1)$, and ends at the point (a,b) inside the first quadrant, and being reflected when it hits the x-axis (at the point $(x,0)$, say).

Show that the two line segments forming its path make equal angles with the x-axis, i.e., "the angle of incidence equals the angle of reflection."

2C-10 A swimmer is on the beach at a point A. The closest point on the straight shoreline to A is called P. There is a platform in the water at B, and the nearest pont on the shoreline to B is called Q. Suppose that the distance from A to P is 100 meters, the distance from B to Q is 100 meters and the distance from P to Q is a meters. Finally suppose that the swimmer can run at 5 meters per second on the beach and swim at 2 meters per second in the water.



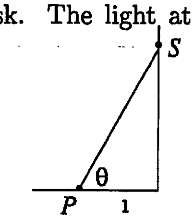
Show that the path the swimmer should take to get to the platform in the least time has the property that the ratio of the sines of the angles the path makes with the shoreline is the reciprocal of the ratio of the speeds in the two regions:

$$\frac{\sin\alpha}{\sin\beta} = \frac{5}{2}$$

(In optics, this is known as Snell's law describing the path taken by a light ray through two successive media. Snell discovered experimentally that the above ratio of sines was a constant, not depending on the starting point and endpoint of the path. This problem shows that Snell's law follows from a minimum principle: the light ray takes the path minimizing its total travel time. The ratio of sines is constant since it depends only on the speeds of light in the two media.)

2C-11 A beam with a rectangular cross-section is cut from a log with a circular cross-section. The strength S of the beam is proportional to the horizontal dimension x of the rectangle and to the cube of the vertical dimension y of the rectangle, $S = cxy^3$. Find the ratio y/x which gives the strongest beam.

2C-12 You are going to mount a light on the wall behind your desk. The light at S illuminates a point P on the horizontal surface of the desk with an intensity inversely proportional to the square of the distance from P to S and proportional to sine of the angle between the ray from P to S and the horizontal surface. Fix a point P on the desk 1 foot from the wall. Find the height of S above the desk for which the intensity at P is largest.



2C-13 a) An airline will fill 100 seats of its aircraft at a fare of \$200. For every \$5 increase in the fare, the plane loses two passengers. For every decrease of \$5, the company gains two passengers. What price maximizes revenue?

b) A utility company has a small power plant that can produce x kilowatt hours of electricity daily at a cost of $10 - x/10^5$ cents each for $0 \le x \le 8 \times 10^5$. Consumers will use $10^5(10 - p/2)$ kilowatt hours of electricity daily at a price of p cents per kilowatt hour. What price should the utility charge to maximize its profit?

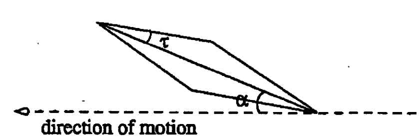
2C-14 Find the maximum value and the location of the maximum for the following functions. (When $a > 0$, $x^a \ln x \to 0$ as $x \to 0^+$. This follows from 2G-8 or from L'Hospital's rule in Section 6A.)

a) $x^2 \ln(1/x)$, for $x > 0$ b) $-x\ln(2x)$, for $x > 0$

2C-15 Find the minimum of $(x+1)e^{-x}$, for $0 \le x < \infty$.

2D. More Max-min Problems

2D-1* Consider a supersonic airplane wing with a cross-section in the shape of a thin diamond (rhombus) in which the half-angle of the opening is τ and the attack angle α. (The attack angle is the angle that the long diagonal of the rhombus makes with the horizontal direction of motion of the plane. See the picture.) The ratio of the lift to the drag is given by the formula (from the Course 16 Unified Engineering notes on aerodynamics):



$$\frac{\text{lift}}{\text{drag}} = \frac{\alpha}{\alpha^2 + \tau^2}$$

a) For a given fixed τ, find the best attack angle α, that is, the one the maximizes the ratio of lift to drag.

b) Find the minimum (largest negative) ratio. (This attack angle could be used in the design of a winged car attempting to break the sound barrier, to prevent it from flying.)

2D-2* Consider a paper cup in the shape of a cone obtained by rotating the line segment $y = ax$, $0 \le x \le r$, around the y axis. For which slope $a \ge 0$ will the paper cup hold the most water, assuming its surface area A is held fixed.

(Use the formulas: volume $V = \frac{1}{3}$(area of the base)(height); $A = \pi r^2\sqrt{1+a^2}$.)

2D-3 Coffee in a cup at a temperature $y(t_0)$ at time t_0 in a room at temperature a cools according to the formula (derived in 3F-4); assume $a = 20°C$ and $c = 1/10$:

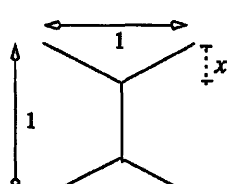
$$y(t) = (y(t_0) - a)e^{-c(t-t_0)} + a, \quad t \geq t_0$$

You are going to add milk so that the cup has 10% milk and 90% coffee. If the coffee has temperature T_1 and the milk T_2, the temperature of the mixture will be $\frac{9}{10}T_1 + \frac{1}{10}T_2$.

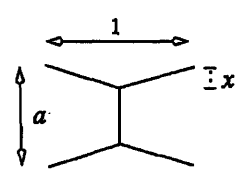
The coffee temperature is 100°C at time $t = 0$, and you will drink the mixture at time $t = 10$. The milk is refrigerated at 5°C. What is the best moment to add the milk so that the coffee will be hottest when you drink it?

2D-4* a) Show that the shortest collection of roads joining four towns at four corners of a unit square is given by roads that meet at 120° angles. Use the variable x as indicated on the picture.

b) Find the shortest collection of roads in the shape indicated for towns at the four corners of a rectangle. Write down the formula for the length of the roads as a function of a. Hint: Sometimes the answer is that the roads meet at 120° angles, but only for certain values of a.



10a



10b

2D-5* Find the triangle of smallest area in the half-plane to the right of the y-axis whose three sides are respectively segments of the x-axis, the diagonal $y = x$, and a line through $(2, 1)$.

2D-6* Find the point of the ellipse

$$73x^2 - 72xy + 52y^2 = 2500$$

closest to $(0, 0)$.

Hint: Use implicit differentiation to find a quadratic equation in x and y. This is the type of equation (known as a homogeneous equation) that you faced in Problem 2C-11. The homogeneous form makes it possible to solve for y/x. You can then find (x, y) by substituting in the original equation.

2D-7* Find two positive numbers whose product is 10 and whose sum is as large as possible.

2E. Related Rates

2E-1 A robot going 20 ft/sec passes under a street light that is 30 feet above the ground. If the robot is 5 feet tall, how fast is the tip of its shadow moving two seconds after passing under the street light? How fast is the length of the shadow increasing at that moment?

2E-2 A beacon light 4 miles offshore (measured perpendicularly from a straight shoreline) is rotating at 3 revolutions per minute. How fast is the spot of light on the shoreline moving when the beam makes an angle of 60° with the shoreline?

2E-3 Two boats are travelling at 30 miles/hr, the first going north and the second going east. The second crosses the path of the first 10 minutes after the first one was there. At what rate is their distance increasing when the second has gone 10 miles beyond the crossing point?

2E-4 Sand is pouring on a conical pile at a rate of 12 m^3 per minute, in such a way that the diameter of the base of the pile is always 3/2 the height. Find the rate at which the height is increasing when the pile is 2 m tall.

2E-5 A person walks away from a pulley pulling a rope slung over it. The rope is being held at a height 10 feet below the pulley. Suppose that the weight at the opposite end of the rope is rising at 4 feet per second. At what rate is the person walking when s/he is 20 feet from being directly under the pulley?

2E-6 An airplane passes directly over a boat at a height of 2 miles. The plane is going north at 400 mph and does not change its altitude. The boat is going west at 50 mph. How rapidly is their distance from each other increasing after one hour?

2E-7 A trough is filled with water at a rate of 1 cubic meter per second. The trough has a trapezoidal cross section with the lower base of length half a meter and one meter sides opening outwards at an angle of 45° from the base. The length of the trough is 4 meters. What is the rate at which the water level h is rising when h is one half meter?

2E-8 One ship is sailing east at 60km an hour and another is sailing south at 50km an hour. The slower ship crosses the path of the faster ship at noon when the faster ship was there one hour earlier. Find the time at which the two ships were closest to each other.

2E-9 A girl slides down a slide in the shape of the parabola $y = (x-1)^2$ for $0 \le x \le 1$. Her vertical speed is $dy/dt = -y(1-y)$. Find her horizontal speed dx/dt when $y = 1/2$.

2E-10 Oil spreads on a frying pan so that its radius is proportional to $t^{1/2}$, where t represents the time from the moment when the oil is poured. Find the rate of change dT/dt of the thickness T of the oil.

2F. Locating zeros; Newton's method

2F-1 a) Graph the function $y = \cos x - x$. Show using y' that there is exactly one root to the equation $\cos x = x$, and give upper and lower bounds on the root.

b) Use Newton's method to find the root to 3 decimal places.

c) Another way to find the root of $\cos x = x$ is to use what is called the fixed point method. Starting with the value $z_1 = 1$, press the cosine key on your calculator until the answer stabilizes, i.e., until $z_{n+1} = \cos z_n$. How many iterations do you need until the first nine digits stabilize? Which method takes fewer steps?

2F-2 Graph the function

$$y = 2x - 4 + \frac{1}{(x-1)^2} \qquad -\infty < x < \infty$$

In particular, count how many times y vanishes. (In math jargon, "vanishes" means $y = 0$, not $y \to \infty$. Another name for the values of x at which $y = 0$ is "the zeros" of y.) Give reasons, based on the sign of y', that the zeros you have found must be there and that there cannot be any more. (For example, "There must be exactly one zero in the interval $(-\infty, -1)$ because ....")

2F-3 Same problem as 2 for $y = x^2 + x^{-1}$

2F-4 The equation $x^5 - x = x(x^2 - 1)(x^2 + 1) = 0$ has three roots, $x = 0, 1, -1$. How many roots does the equation $x^5 - x - 1/2 = 0$ have?

2F-5 a) Find an initial value x_1 for the zero of $x - x^3 = 0$ for which Newton's method gives an undefined quantity for x_2.

b) Find an initial value x_1 for the zero of $x - x^3 = 0$ such that Newton's method bounces back and forth between two values forever. Hint: use symmetry.

c) Find the largest interval around each of the roots of $x - x^3 = 0$ such that Newton's method converges to that root for every initial value x_1 in the interval. Hint: Parts (a) and (b) should help.

2F-6 a) Suppose that a company manufacturing cylindrical beakers (of uniform thickness on the bottom and the sides) is willing to use up to 10 percent more glass than the minimum required to hold a particular volume. What proportions are permitted? (You will need to use Newton's method.)

b) What is the connection with Problem 2C-5?

2F-7 Find the point of the curve $y = \cos x$ closest to the origin. (Minimize the square of the distance to the origin; prove you've found all the critical points.)

2G. Mean-value Theorem

2G-1 For each of these functions, on the indicated interval, find explicitly the point c whose existence is predicted by the Mean-value Theorem; if there is more than one such c, find all of them. Use the form (1).

(a) x^2 on $[0,1]$ (b) $\ln x$ on $[1,2]$ (c) $x^3 - x$ on $[-2,2]$

2G-2 Using the form (2), show that

(a) $\sin x < x$, if $x > 0$ (b) $\sqrt{1+x} < 1 + x/2$ if $x > 0$.

2G-3 The Mass Turnpike is 121 miles long. An SUV enters at the Boston end at noon and and emerges at the west end at 1:50. Prove that at some moment during the trip it was speeding (i.e., over the 65 mph limit).

2G-4 A polynomial $p(x)$ of degree n has at most n distinct real roots, but it may have fewer — for instance, $x^2 + 1$ has no real roots at all. However, show that if $p(x)$ does have n distinct real roots, then $p'(x)$ has $n - 1$ distinct real roots.

2G-5 a) Suppose $f''(x)$ exists on an interval I and $f(x)$ has a zero at three distinct points $a < b < c$ on I. Show there is a point p on $[a, c]$ where $f''(p) = 0$.

b) Illustrate part (a) on the cubic $f(x) = (x - a)(x - b)(x - c)$.

2G-6 Using the form (2) of the Mean-value Theorem, prove that on an interval $[a, b]$,

a) $f'(x) > 0 \quad \Rightarrow \quad f(x)$ increasing; (b) $f'(x) = 0 \quad \Rightarrow \quad f(x)$ constant.

2G-7* In what follows, use the following consequence of the Mean-value Theorem:

if $f(a) \geq g(a)$ and $f'(x) > g'(x)$ for all $x > a$, then $f(x) > g(x)$ for all $x > a$.

a) Starting with $e^x > 0$, show that for all $x > 0$, we have $e^x > 1$.

b) Use the same method as in part (a) to show that $e^x > 1 + x$, and then from this deduce that $e^x > 1 + x + x^2/2$.

Remark This process can be continued. In the limit it leads to the infinite series which represents e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

This infinite series can be used to define e^x, and it is a good way to compute it to high accuracy.

c) Show that for each number $n > 0$

$$nx^{1/n} < \ln x \quad \text{for } x > 1$$

2G-8* An analogous principle to the one in 2G-7 can be used if $x < a$:

if $f(a) \geq g(a)$ and $f'(x) < g'(x)$ for all $x < a$, then $f(x) > g(x)$ for all $x < a$.

(Why did the inequalities get reversed when $x < a$? Draw a graph to see.)

Use this principle with $f(x) = \ln x$ and $a = 1$ to show that

$$\ln x > -nx^{-1/n} \quad \text{for } 0 < x < 1,\ n > 0.$$

Unit 3. Integration

3A. Differentials, indefinite integration

3A-1 Compute the differentials $df(x)$ of the following functions.

a) $d(x^7 + \sin 1)$ b) $d\sqrt{x}$ c) $d(x^{10} - 8x + 6)$

d) $d(e^{3x} \sin x)$ e) Express dy in terms of x and dx if $\sqrt{x} + \sqrt{y} = 1$

3A-2 Compute the following indefinite integrals

a) $\int (2x^4 + 3x^2 + x + 8)dx$ b) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$ c) $\int \sqrt{8+9x}dx$

d) $\int x^3(1 - 12x^4)^{1/8}dx$ e) $\int \frac{x}{\sqrt{8 - 2x^2}}dx$ f) $\int e^{7x}dx$

g) $\int 7x^4 e^{x^5} dx$ h) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$ i) $\int \frac{dx}{3x+2}$

j) $\int \frac{x+5}{x}dx.$ k) $\int \frac{x}{x+5}dx.$ (Write $\frac{x}{x+5} = 1 + \ldots.$)

l) $\int \frac{\ln x}{x}dx$ m) $\int \frac{dx}{x \ln x}$

3A-3 Compute the following indefinite integrals.

a) $\int \sin(5x)dx$ b) $\int \sin(x)\cos(x)dx$ c) $\int \cos^2 x \sin x dx$

d) $\int \frac{\cos x}{\sin^3 x}dx$ e) $\int \sec^2(x/5)dx$ f) $\int \tan^6 x \sec^2 x dx$

g) $\int \sec^9 x \tan x dx$

3B. Definite Integrals

3B-1 Evaluate

a) $\sum_{n=1}^{4} n^2$ b) $\sum_{j=1}^{6} 2^j$

c) $\sum_{j=1}^{5} (-1)^j j^2$ d) $\sum_{n=1}^{4} \frac{1}{n}$

3B-2 Find a Σ notation expression for

a) $3 - 5 + 7 - 9 + 11 - 13$ b) $1 + 1/4 + 1/9 + \cdots + 1/n^2$

c) $\sin x/n + \sin(2x/n) + \cdots + \sin((n-1)x/n) + \sin x$

3B-3 Write the upper, lower, left and right Riemann sums for the following integrals, using 4 equal subintervals:

a) $\int_0^1 x^3 dx$ b) $\int_{-1}^{3} x^2 dx$ c) $\int_0^{2\pi} \sin x dx$

3B-4 Calculate the difference between the upper and lower Riemann sums for the following integrals with n intervals

a) $\int_0^b x^2 dx$ b) $\int_0^b x^3 dx$

Does the difference tend to zero as n tends to infinity?

3B-5 Evaluate the limit, by relating it to a Riemann sum.

$$\lim_{n\to\infty} \frac{\sin(b/n) + \sin(2b/n) + \cdots + \sin((n-1)b/n) + \sin(nb/n)}{n}$$

3B-6* Calculate $\int_0^1 e^x dx$ by using upper Riemann sums.

Hints: The sum is a geometric progression. You will need the limit $\lim_{n\to\infty} n(e^{1/n} - 1)$. This can be evaluated putting $h = 1/n$ and relating the limit to the derivative of e^x at $x = 0$.

3B-7* Evaluate the limit

$$\lim_{n\to\infty} \frac{2^{b/n} + 2^{2b/n} + \ldots + 2^{(n-1)b/n} + 2^{nb/n}}{n}$$

(See **3B-6**.)

3C. Fundamental theorem of calculus

3C-1 Find the area under the graph of $y = 1/\sqrt{x-2}$ for $3 \le x \le 6$

3C-2 Calculate

a) $\int_0^2 \sqrt{3x+5}dx$ b) $\int_0^2 (3x+5)^n dx$ c) $\int_{3\pi/4}^{\pi} \frac{\sin x dx}{\cos^3 x}$

3C-3 Calculate

a) $\int_1^2 \frac{x dx}{x^2+1}$ b) $\int_b^{2b} \frac{x dx}{x^2+b^2}$

3C-4 Calculate $\lim_{b\to\infty} \int_1^b x^{-10} dx$. What area does this integral describe?

3C-5 Find the area

a) under one arch of $\sin x$. b) under one arch of $\sin ax$ for a positive constant a.

3C-6 Find the area between the x-axis and

a) the curve $y = x^2 - 4$ b) the curve $y = x^2 - a$ for $a > 0$.

3D. Second fundamental theorem

3D-1 a) Prove that $\displaystyle\int_0^x \frac{dt}{\sqrt{t^2+a^2}} = \ln(x+\sqrt{x^2+a^2}) - \ln a, a>0, x>0.$

b) For what c is $\displaystyle\int_c^x \frac{dt}{\sqrt{t^2+a^2}} = \ln(x+\sqrt{x^2+a^2})$?

3D-2* Show that the function $y = \displaystyle\int_0^x \sqrt{1-t^2}\,dt$ satisfies the differential equation with side conditions:

$$y'y'' = -x; \quad y(0)=0,\, y'(0)=1.$$

3D-3 Discuss the function $F(x) = \displaystyle\int_0^x \frac{1-t^2}{1+t^2}dt$, including a sketch; describe

a) domain

b) relative maxima and minima, where increasing or decreasing, points of inflection

c) behavior as $x \to \infty$ (Hint: Evaluate the integrand as $t \to \pm\infty$.)

d) symmetry about y-axis or origin

3D-4 Find a function whose derivative is $\sin(x^3)$ and whose value at 0 is

a) 0 b) 2 c) find one whose value at 1 is -1

3D-5 Evaluate $\displaystyle\lim_{\Delta x\to 0} \frac{1}{\Delta x}\int_1^{1+\Delta x} \frac{t}{\sqrt{1+t^4}}dt$ two ways:

a) by interpreting the integral as the area under a curve

b) by relation the limit to $F'(1)$, where $F(x) = \displaystyle\int_0^x \frac{t}{\sqrt{1+t^4}}dt$

3D-6 For different values of a, the functions $F(x) = \displaystyle\int_a^x dt$ differ from each other by constants. Show this two ways:

a) directly

b) using the corollary to the mean-value theorem quoted ((8), p.FT.5)

3D-7 Evaluate $F'(x)$ if $F(x) =$

a) $\displaystyle\int_0^{x^2} \sqrt{u}\sin u\,du$ b) $\displaystyle\int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}$

c) $\displaystyle\int_x^{x^2} \tan u\,du$

3D-8 Let $f(x)$ be continuous. Find $f(\pi/2)$ if :

a) $\displaystyle\int_0^x f(t)dt = 2x(\sin x+1)$ b) $\displaystyle\int_0^{x/2} f(t)dt = 2x(\sin x+1)$

3E. Change of variables; Estimating integrals

3E-1 Prove directly from the definition $L(x) = \int_1^x \frac{dt}{t}$ that $L(\frac{1}{a}) = -L(a)$, by making a change of variables in the definite integral.

3E-2 The function defined by $E(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-u^2/2} du$ is used in probability and statistics and has the same importance as sine and cosine functions have to trigonometry.

a) Express $E(x)$ in terms of the function of example 5 of Notes FT, $F(x) = \int_0^x e^{-t^2} dt$, by making a change of variable. It is known that $\lim_{x\to\infty} F(x) = \sqrt{\pi}/2$. What is $\lim_{x\to\infty} E(x)$?

b) Evaluate $\lim_{N\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} e^{-u^2/2} du$ and $\lim_{x\to-\infty} E(x)$.

c) Express $\frac{1}{\sqrt{2\pi}} \int_a^b e^{-u^2/2} du$ in terms of the function E and the constants a and b, where $a < b$.

3E-3 Evaluate by making a substitution and changing both the variable and the limits of integration.

a) $\int_1^e \frac{\sqrt{\ln x}}{x} dx$ $(u = \ln x)$

b) $\int_0^\pi \frac{\sin x}{(2+\cos x)^3} dx$

c) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ $(x = \sin u)$

d) $\int_{18}^{19} \frac{dx}{x^2 - 34x + 289}$ $(x = z + 17)$

3E-4 From the definite integral $\int_{-1}^{1} \sqrt{1-x^2} dx = \pi/2$ deduce the value of $\int_{-a}^{a} \sqrt{a^2-x^2} dx$ by making a suitable change of variable of the form $x = ct$ (c constant).

3E-5 Let $F(x) = \int_0^x f(t) dt$.

a) Prove that if $f(t)$ is even, then $F(x)$ is odd.

b) Prove that if $f(t)$ is odd, then $F(x)$ is even.

Hint: Make the change of variable $u = -t$ in the definite integral. (Compare with **4-6** .)

3E-6 By comparing the given integral with an integral that is easier to evaluate. establish each of the following estimations:

a) $\int_0^1 \frac{dx}{1+x^3} > 0.65$

b) $\int_0^\pi \sin^2 x \, dx < 2$

c) $\int_{10}^{20} \sqrt{x^2+1} \, dx > 150$

3E-7 Show $\left|\int_1^N \frac{\sin x}{x^2}dx\right| < 1$

3F. Differential equations: separation of variables

3F-1 Solve the following differential equations

a) $dy/dx = (2x+5)^4$ b) $dy/dx = (y+1)^{-1}$

c) $dy/dx = 3/\sqrt{y}$ d) $dy/dx = xy^2$

3F-2 Solve each differential equation with the given initial condition, and evaluate the solution at the given value of x:

a) $dy/dx = 4xy$, $y(1) = 3$. Find $y(3)$.

b) $dy/dx = \sqrt{y+1}$, $y(0) = 1$. Find $y(3)$.

c) $dy/dx = x^2y^{-1}$, $y(0) = 10$. Find $y(5)$.

d) $dy/dx = 3y+2$, $y(0) = 0$. Find $y(8)$.

e) $dy/dx = e^y$, $y(3) = 0$. Find $y(0)$. For which values of x is the solution y defined?

3F-3 a) Solve $dy/dx = y^2$ with $y = 1$ at $x = 0$. Evaluate y at $x = 1/2$, at $x = -1$, and at $x = 1$.

b) Graph the solution and use the graph to discuss the range of validity of the formula for y. In particular, explain why the apparent value at $x = 3/2$ is suspect.

3F-4 Newton's law of cooling says that the rate of change of temperature is proportional to the temperature difference. In symbols, if a body is at a temperature T at time t and the surrounding region is at a constant temperature T_e (e for external), then the rate of change of T is given by

$$dT/dt = k(T_e - T).$$

The constant $k > 0$ is a constant of proportionality that depends properties of the body like specific heat and surface area.

a) Why is $k > 0$ the only physically realistic choice?

b) Find the formula for T if the initial temperature at time $t = 0$ is T_0.

c) Show that $T \to T_e$ as $t \to \infty$.

d) Suppose that an ingot leaves the forge at a temperature of 680° Celsius in a room at 40° Celsius. It cools to 200° in eight hours. How many hours does it take to cool from 680° to 50°? (It is simplest to keep track of the temperature difference $T - T_e$, rather than T. The temperature difference undergoes exponential decay.)

e) Suppose that an ingot at 1000° cools to 800° in one hour and to 700° in two hours. Find the temperature of the surrounding air.

f) Show that $y(t) = T(t - t_0)$ also satisfies Newton's law of cooling for any constant t_0. Write out the formula for $T(t-t_0)$ and show that it is the same as the formula in E10/17 for $y(t)$ by identifying the constants k, T_e and T_0 with their corresponding values in the displayed formula in E10/17.

3F-5* Air pressure satisfies the differential equation $dp/dh = -(.13)p$, where h is the altitude from sea level measured in kilometers.

a) At sea level the pressure is[1] $1\,\text{kg/cm}^2$. Solve the equation and find the pressure at the top of Mt. Everest (10 km).

b) Find the difference in pressure between the top and bottom of the Green Building. (Pretend it's 100 meters tall starting at sea level.) Compute the numerical value using a calculator. Then use instead the linear approximation to e^x near $x = 0$ to estimate the percentage drop in pressure from the bottom to the top of the Green Building.

c) Use the linear approximation $\Delta p \approx p'(0)\Delta h$ and compute $p'(0)$ directly from the differential equation to find the drop in pressure from the bottom to top of the Green Building. Notice that this gives an answer without even knowing the solution to the differential equation. Compare with the approximation in part (b). What does the linear approximation $p'(0)\Delta h$ give for the pressure at the top of Mt. Everest?

d) What is the differential equation for p if altitude is measured in meters instead of kilometers?

3F-6 Let $y = \cos^3 u - 3\cos u$, $x = \sin^4 u$. Find dy, dx, and dy/dx. Simplify.

3F-7 Solve:
a) $y' = -xy, \quad y(0) = 1$ b) $\cos x \sin y\, dy = \sin x\, dx; \quad y(0) = 0$.

3F-8 a) Find all plane curves such that the tangent line at P intersects the x-axis 1 unit to the left of the projection of P on the the x-axis.

b) Find all plane curves in the first quadrant such that for every point P on the curve, P bisects the part of the tangent line at P that lies in the first quadrant.

3G. Numerical Integration

3G-1 Find approximations to the following integrals using four intervals using Riemann sums with left endpoints, using the trapezoidal rule, and using Simpson's rule. Also give numerical approximations to the exact values of the integrals given to see how good these approximation methods are.

a) $\displaystyle\int_0^1 \sqrt{x}dx \ (= 2/3.)$ b) $\displaystyle\int_0^\pi \sin x dx \ (= 2.)$

c) $\displaystyle\int_0^1 \frac{dx}{1+x^2} \ (= \pi/4;$ cf. unit 5) d) $\displaystyle\int_1^2 \frac{dx}{x} \ (= \ln 2)$

3G-2 Show that the value given by Simpson's rule for two intervals for the integral

$$\int_0^b f(x)dx$$

gives the exact answer when $f(x) = x^3$. (Since a cubic polynomial is a sum of a quadratic polynomial and a polynomial ax^3, and Simpson's rule is exact for any quadratic polynomial, the result of this exercise implies by linearity (cf. Notes PI) that Simpson's rule will also be exact for any cubic polynomial.)

[1] using the correspondence between weight and mass on Earth of $F = ma$ with $a = 10m/sec^2$

3G-3 Use the trapezoidal rule to estimate $\sqrt{1}+\sqrt{2}+\sqrt{3}+...+\sqrt{10,000}$. Is your estimate too high or too low?

3G-4 Use the trapezoidal rule to estimate the sum of the reciprocals of the first n integers. Is your estimate too high or too low?

3G-5 If the trapezoidal rule is used to estimate the value of $\int_a^b f(x)dx$ under what hypotheses on $f(x)$ will the estimate be too low? too high?

Unit 4. Applications of integration

4A. Areas between curves.

4A-1 Find the area between the following curves

a) $y = 2x^2$ and $y = 3x - 1$ b) $y = x^3$ and $y = ax$; assume $a > 0$

c) $y = x + 1/x$ and $y = 5/2$. d) $x = y^2 - y$ and the y axis.

4A-2 Find the area under the curve $y = 1 - x^2$ in two ways.

4A-3 Find the area between the curves $y = 4 - x^2$ and $y = 3x$ in two ways.

4A-4 Find the area between $y = \sin x$ and $y = \cos x$ from one crossing to the next.

4B. Volumes by slicing; volumes of revolution.

4B-1 Find the volume of the solid of revolution generated by rotating the regions bounded by the curves given around the x-axis.

a) $y = 1 - x^2$, $y = 0$ b) $y = a^2 - x^2$, $y = 0$ c) $y = x$, $y = 0$, $x = 1$

d) $y = x$, $y = 0$, $x = a$ e) $y = 2x - x^2$, $y = 0$ f) $y = 2ax - x^2$, $y = 0$

g) $y = \sqrt{ax}$, $y = 0$, $x = a$ h) $x^2/a^2 + y^2/b^2 = 1$, $x = 0$

4B-2 Find the volume of the solid of revolution generated by rotating the regions in 4B-1 around the y-axis.

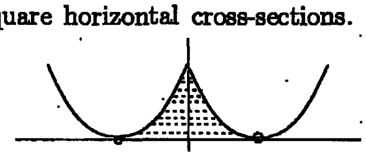
4B-3 Show that the volume of a pyramid with a rectangular base is $bh/3$, where b is the area of the base and h is the height. (Show in the process that the proportions of the rectangle do not matter.)

4B-4 Consider (x, y, z) such that $x^2 + y^2 < 1$, $x > 0$ and $0 \le z \le 5$. This describes one half of cylinder (a split log). Chop out a wedge out of the log along $z = 2x$. Find the volume of the wedge.

4B-5 Find the volume of the solid obtained by revolving an equilateral triangle of sidelength a around one of its sides.

4B-6 The base of a solid is the disk $x^2 + y^2 \le a^2$. Planes perpendicular to the xy-plane and perpendicular to the x-axis slice the solid in isoceles right triangles. The hypotenuse of these trianglesis the segment where the plane meets the disk. What is the volume of the solid?

4B-7 A tower is constructed with a square base and square horizontal cross-sections. Viewed from any direction perpendicular to a side, the tower has base $y = 0$ and profile lines $y = (x - 1)^2$ and $y = (x + 1)^2$. (See shaded region in picture.) Find the volume of the solid.



4C. Volumes by shells

4C-1 Assume that $0 < a < b$. Revolve the disk $(x-b)^2 + y^2 \le a^2$ around the y axis. This doughnut shape is known as a torus.

a) Set up the integral for volume using integration dx

b) Set up the integral for volume using integration dy

c) Evaluate (b).

d) (optional) Show that the (a) and (b) are the same using the substitution $z = x - b$.

4C-2 Find the volume of the region $0 \le y \le x^2$, $x \le 1$ revolved around the y-axis.

4C-3 Find the volume of the region $\sqrt{x} \le y \le 1$, $x \ge 0$ revolved around the y-axis by both the method of shells and the method of disks and washers.

4C-4 Set up the integrals for the volumes of the regions in 4B-1 by the method of shells. (Do not evaluate.)

4C-5 Set up the integrals for the volumes of the regions in 4B-2 by the method of shells. (Do not evaluate.)

4C-6 Let $0 < a < b$. Consider a ball of radius b and a cylinder of radius a whose axis passes through the center of the ball. Find the volume of the ball with the cylinder removed.

4D. Average value

4D-1 What is the average cross-sectional area of the solid obtained by revolving the region bounded by $x = 2$, the x-axis, and the curve $y = x^2$ about the x-axis? (Cross-sections are taken perpendicular to the x-axis.)

4D-2 Show that the average value of $1/x$ over the interval $[a, 2a]$ is of the form C/a, where C is a constant independent of a. (Assume $a > 0$.)

4D-3 A point is moving along the x-axis, with distance function given by $x = s(t)$. Show that over a time interval $[a, b]$, the average value of its velocity $v(t)$ is the same as its average velocity over this interval.

4D-4 What is the average value of the square of the distance of a point P from a fixed point Q on the unit circle, where P is chosen at random on the circle? (Use coordinates; place Q on the x-axis.) Check your answer for reasonableness.

4D-5 If the average value of $f(t)$ between 0 and x is given by the function $g(x)$, express $f(x)$ in terms of $g(x)$.

4D-6 An amount of money A compounded continuously at interest rate r increases according to the law

$$A(t) = A_0 e^{rt} \qquad (t = \text{time in years})$$

a) What is the average amount of money in the bank over the course of T years?

b) Suppose r and T are small. Give an approximate answer to part (a) by using the quadratic approximation to your exact answer; check it for reasonableness.

4D-7 Find the average value of x^2 in $0 \le x \le b$.

4D-8 Find the average distance from a point on the perimeter of a square of sidelength a to the center. Find the average of the square of the distance.

4D-9 Find the average value of $\sin ax$ in its first hump.

4E. Parametric equations

4E-1 Find the rectangular equation for $x = t + t^2$, $y = t + 2t^2$.

4E-2 Find the rectangular equation for $x = t + 1/t$ and $y = t - 1/t$ (compute x^2 and y^2).

4E-3 Find the rectangular equation for $x = 1 + \sin t$, $y = 4 + \cos t$.

4E-4 Find the rectangular equation for $x = \tan t$, $y = \sec t$.

4E-5 Find the rectangular equation for $x = \sin 2t$, $y = \cos t$.

4E-6 Consider the parabola $y = x^2$. Find the parametrization using the slope of the curve at a point (x, y) as the parameter.

4E-7 Find the parametrization of the circle $x^2 + y^2 = a^2$ using the slope as the parameter. Which portion of the circle do you obtain in this way?

4E-8 At noon, a snail starts at the center of an open clock face. It creeps at a steady rate along the hour hand, reaching the end of the hand at 1:00 PM. The hour hand is 1 meter long. Write parametric equations for the position of the snail at time t, in some reasonable xy-coordinate system.

4E-9* a) What part of a train is moving backwards when the train moves forwards?

b) A circular disc has inner radius a and outer radius b. Its inner circle rolls along the positive x-axis without slipping . Find parametric equations for the motion of a point P on its outer edge, assuming P starts at $(0, b)$. Use θ as parameter. (Your equations should reduce to those of the cycloid when $a = b$. Do they?)

c) Sketch the curve that P traces out.

d) Show from the parametric equations you found that P is moving backwards whenever it lies below the x-axis.

4F. Arclength

4F-1 Find the arclength of the following curves

a) $y = 5x + 2$, $0 \le x \le 1$. b) $y = x^{3/2}$, $0 \le x \le 1$.

c) $y = (1 - x^{2/3})^{3/2}$, $0 \le x \le 1$. d) $y = (1/3)(2 + x^2)^{3/2}$, $1 \le x \le 2$.

4F-2 Find the length of the curve $y = (e^x + e^{-x})/2$ for $0 \le x \le b$. Hint:

$$\left(\frac{e^x - e^{-x}}{2}\right)^2 + 1 = \left(\frac{e^x + e^{-x}}{2}\right)^2$$

4F-3 Express the length of the parabola $y = x^2$ for $0 \le x \le b$ as an integral. (Do not evaluate.)

4F-4 Find the length of the curve $x = t^2$, $y = t^3$ for $0 \le t \le 2$.

4F-5 Find an integral for the length of the curve given parametrically in Exercise 4E-2 for $1 \le t \le 2$. Simplify the integrand as much as possible but do not evaluate.

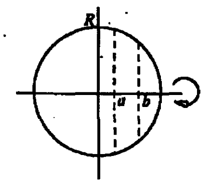
4F-6 a) The cycloid given parametrically by $x = t - \sin t$, $y = 1 - \cos t$ describes the path of a point on a rolling wheel. If t represents time, then the wheel is rotating at a constant speed. How fast is the point moving at each time t? When is the forward motion (dx/dt) largest and when is it smallest?

b) Find the length of the cycloid for one turn of the wheel. (Use a half angle formula.)

4F-7 Express the length of the ellipse $x^2/a^2 + y^2/b^2 = 1$ using the parametrization $x = a\cos t$ and $y = b \sin t$. (Do not evaluate.)

4F-8 Find the length of the curve $x = e^t \cos t$, $y = e^t \sin t$ for $0 \le t \le 10$.

4G. Surface Area



4G-1 Consider the sphere of radius R formed by revolving the circle $x^2 + y^2 = R^2$ around the x-axis. Show that for $-R \le a < b \le R$, the portion of the sphere $a \le x \le b$ has surface area $2\pi R(b - a)$. For example, the hemisphere, $a = 0$, $b = R$ has area $2\pi R^2$.

4G-2 Find the area of the segment of $y = 1 - 2x$ in the first quadrant revolved around the x-axis.

4G-3 Find the area of the segment of $y = 1 - 2x$ in the first quadrant revolved around the y-axis.

4G-4 Find an integral formula for the area of $y = x^2$, $0 \le x \le 4$ revolved around the x-axis. (Do not evaluate.)

4G-5 Find the area of $y = x^2$, $0 \le x \le 4$ revolved around the y-axis.

4G-6 Find the area of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ revolved around the x-axis.

4G-7 Conside the torus of Problem E22/1.

a) Set up the integral for surface area using integration dx

b) Set up the integral for surface area using integration dy

c) Evaluate (b) using the substitution $y = a \sin\theta$.

4H. Polar coordinate graphs

4H-1 For each of the following points given in rectangular coordinates, give its polar coordinates. (For points below the x-axis, give two expressions for its polar coordinates, using respectively positive and negative values for θ.)

a) $(0, 3)$	b) $(-2, 0)$	c) $(1, \sqrt{3})$	d) $(-2, 2)$
e) $(1, -1)$	f) $(0, -2)$	g) $(\sqrt{3}, -1)$	h) $(-2, -2)$

4H-2

a) Find using two different methods the equation in polar coordinates for the circle of radius a with center at $(a, 0)$ on the x-axis, as follows:

(i) write its equation in rectangular coordinates, and then change it to polar coordinates (substitute $x = r\cos\theta$ and $y = r\sin\theta$, and then simplify).

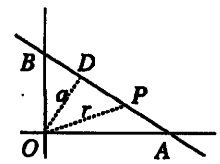
(ii) treat it as a locus problem: let OQ be the diameter lying along the x-axis, and $P : (r, \theta)$ a point on the circle; use ΔOPQ and trigonometry to find the relation connecting r and θ.

b) Carry out the analogue of 4H-2a for the circle of radius a with center at $(0, a)$ on the y-axis; OQ is now the diameter lying along the y-axis.

c) (i) Find the polar equation for the line intersecting the positive x- and y-axes respectively at A and B, and having perpendicular distance a from the origin.

(Let $\alpha = \angle DOA$; use the right triangle DOP to get the equation connecting r, θ, α and a.

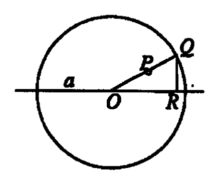
(ii) Convert your polar equation to the usual rectangular equation involving A and B, by using trigonometry.



d) In the accompanying figure, the point Q moves around the circle of radius a centered at the origin; QR is a perpendicular to the x-axis. P is a point on ray OQ such that $|QP| = |QR|$: P is the point inside the circle in the first two quadrants, but outside the circle in the last two quadrants.

(i) Sketch the locus of P; the locus is called a *cardioid* (cf. 4H-3c).

(ii) find the polar equation of this locus.



e) The point P moves in a locus so that the product of its distances from the two points $Q : (-a, 0)$ and $R : (a, 0)$ is constant. Assuming the locus of P goes through the origin, determine the value of the constant, and derive the polar equation of the locus of P.

(Work with the squares of the distances, rather than the distances themselves, and use the law of cosines; the identities $(A+B)(A-B) = A^2 - B^2$ and $\cos 2\theta = 2\cos^2\theta - 1$ simplify the algebra and produce a simple answer at the end. The resulting curve is a *lemniscate*, cf. 4H-3g.)

4H-3 For each of the following,

(i) give the corresponding equation in rectangular coordinates;

(ii) draw the graph; indicate the direction of increasing θ.

a) $r = \sec\theta$ b) $r = 2a\cos\theta$

c) $r = (a + b\cos\theta)$ (This figure is a cardioid for $a = b$, a limaçon with a loop for $0 < a < b$, and a limaçon without a loop for $a > b > 0$.)

d) $r = a/(b + c\cos\theta)$ (Assume the constants a and b are positive. This figure is an ellipse for $b > |c| > 0$, a circle for $c = 0$, a parabola for $b = |c|$, and a hyperbola for $b < |c|$.)

e) $r = a\sin(2\theta)$ (4-leaf rose) f) $r = a\cos(2\theta)$ (4-leaf rose)

g) $r^2 = a^2\sin(2\theta)$ (lemniscate) h) $r^2 = a^2\cos(2\theta)$ (lemniscate)

i) $r = e^{a\theta}$ (logarithmic spiral)

4I. Area and arclength in polar coordinates

4I-1 Find the arclength element $ds = w(\theta)d\theta$ for the curves of 4H-3.

4I-2 Find the area of one leaf of a three-leaf rose $r = a\cos(3\theta)$.

4I-3 Find the area of the region $0 \le r \le e^{3\theta}$ for $0 \le \theta \le \pi$

4I-4 Find the area of one loop of the lemniscate $r^2 = a^2 \sin(2\theta)$

4I-5 What is the average distance of a point on a circle of radius a from a fixed point Q on the circle? (Place the circle so Q is at the origin and use polar coordinates.)

4I-6 What is the average distance from the x-axis of a point chosen at random on the cardioid $r = a(1 - \cos\theta)$, if the point is chosen

a) by letting a ray $\theta = c$ sweep around at uniform velocity, stopping at random and taking the point where it intersects the cardioid;

b) by letting a point P travel around the cardioid at uniform velocity, stopping at random; (the answers to (a) and (b) are different...)

4I-7 Calculate the area and arclength of a circle, parameterized by $x = a\cos\theta, y = a\sin\theta$.

4J. Other Applications

4J-1 Suppose it takes k units of energy to lift a cubic meter of water one meter. About how much energy E will it take to pump dry a circular hole one meter in diameter and 100 meters deep that is filled with water? (Give reasoning.)

4J-2 The amount x (in grams) of a radioactive material declines exponentially over time (in minutes), according to the law $x = x_0 e^{-kt}$, where x_0 is the amount initially present at time $t = 0$. If one gram of the material produces r units of radiation/minute, about how much radiation R is produced over one hour by x_0 grams of the material? (Give reasoning.)

4J-3 A very shallow circular reflecting pool has uniform depth D, and radius R (meters). A disinfecting chemical is released at its center, and after a few hours of symmetrical diffusion outwards, the concentration of chemical at a point r meters from the center is $\dfrac{k}{1+r^2}$ g/m^3.

What amount A of the chemical was released into the pool? (Give reasoning.)

4J-4 Assume a heated outdoor pool requires k units of heat/hour for each degree F it is maintained above the external air temperature.

If the external temperature T varies between 50^o and 70^o over a 24 hour period starting at midnight, according to $T = 10\big(6 - \cos(\pi t/12)\big)$, how many heat units will be required to maintain the pool at a steady 75^o temperature? (Give reasoning.)

4J-5 A manufacturers cost for storing one unit of inventory is c dollars/day for space and insurance. Over the course of 30 days, production P rises from 10 to 40 units/day according to $P = 10 + t$. Assuming no units are sold, what is the inventory cost for this period? (Give reasoning.)

Unit 5. Integration techniques

5A. Inverse trigonometric functions; Hyperbolic functions

5A-1 Evaluate

a) $\tan^{-1}\sqrt{3}$ b) $\sin^{-1}(\sqrt{3}/2)$

c) If $\theta = \tan^{-1} 5$, then evaluate $\sin\theta$, $\cos\theta$, $\cot\theta$, $\csc\theta$, and $\sec\theta$.

d) $\sin^{-1}\cos(\pi/6)$ e) $\tan^{-1}\tan(\pi/3)$

f) $\tan^{-1}\tan(2\pi/3)$ g) $\lim\limits_{x\to-\infty}\tan^{-1}x$.

5A-2 Calculate

a) $\displaystyle\int_1^2 \frac{dx}{x^2+1}$ b) $\displaystyle\int_b^{2b} \frac{dx}{x^2+b^2}$ c) $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$.

5A-3 Calculate the derivative with respect to x of the following

a) $\sin^{-1}\left(\dfrac{x-1}{x+1}\right)$ b) $\tanh x$

c) $\ln(x+\sqrt{x^2+1})$ d) y such that $\cos y = x$, $0 \le x \le 1$ and $0 \le y \le \pi/2$.

e) $\sin^{-1}(x/a)$ f) $\sin^{-1}(a/x)$

g) $\tan^{-1}(x/\sqrt{1-x^2})$ h) $\sin^{-1}\sqrt{1-x}$

5A-4 a) If the tangent line to $y = \cosh x$ at $x = a$ goes through the origin, what equation must a satisfy?

b) Solve for a using Newton's method.

5A-5 a) Sketch the graph of $y = \sinh x$, by finding its critical points, points of inflection, symmetries, and limits as $x \to \infty$ and $-\infty$.

b) Give a suitable definition for $\sinh^{-1} x$, and sketch its graph, indicating the domain of definition. (The inverse hyperbolic sine.)

c) Find $\dfrac{d}{dx}\sinh^{-1} x$.

d) Use your work to evaluate $\displaystyle\int \frac{dx}{\sqrt{a^2+x^2}}$

5A-6 a) Find the average value of y with respect to arclength on the semicircle $x^2+y^2=1$, $y>0$, using polar coordinates.

b) A weighted average of a function is

$$\int_a^b f(x)w(x)dx \Big/ \int_a^b w(x)dx$$

Do part (a) over again expressing arclength as $ds = w(x)dx$. The change of variables needed to evaluate the numerator and denominator will bring back part (a).

c) Find the average height of $\sqrt{1-x^2}$ on $-1 < x < 1$ with respect to dx. Notice that this differs from part (b) in both numerator and denominator.

5B. Integration by direct substitution

Evaluate the following integrals

5B-1. $\int x\sqrt{x^2-1}dx$ 5B-2. $\int e^{8x}dx$ 5B-3. $\int \frac{\ln x dx}{x}$

5B-4. $\int \frac{\cos x dx}{2+3\sin x}$ 5B-5. $\int \sin^2 x \cos x dx$ 5B-6. $\int \sin 7x dx$

5B-7. $\int \frac{6x dx}{\sqrt{x^2+4}}$ 5B-8. $\int \tan 4x dx$ 5B-9. $\int e^x(1+e^x)^{-1/3}dx$

5B-10. $\int \sec 9x dx$ 5B-11. $\int \sec^2 9x dx$ 5B-12. $\int xe^{-x^2}dx$

5B-13. $\int \frac{x^2 dx}{1+x^6}$. Hint: Try $u = x^3$.

Evaluate the following integrals by substitution and changing the limits of integration.

5B-14. $\int_0^{\pi/3} \sin^3 x \cos x dx$ 5B-15. $\int_1^e \frac{(\ln x)^{3/2}dx}{x}$ 5B-16. $\int_{-1}^1 \frac{\tan^{-1} x dx}{1+x^2}$

5C. Trigonometric integrals

Evaluate the following

5C-1. $\int \sin^2 x dx$ 5C-2. $\int \sin^3(x/2)dx$ 5C-3. $\int \sin^4 x dx$

5C-4. $\int \cos^3(3x)dx$ 5C-5. $\int \sin^3 x \cos^2 x dx$ 5C-6. $\int \sec^4 x dx$

5C-7. $\int \sin^2(4x)\cos^2(4x)dx$ 5C-8. $\int \tan^2(ax)\cos(ax)dx$ 5C-9. $\int \sin^3 x \sec^2 x dx$

5C-10. $\int (\tan x + \cot x)^2 dx$ 5C-11. $\int \sin x \cos(2x)dx$ (Use double angle formula.)

5C-12. $\int_0^{\pi} \sin x \cos(2x)dx$ (See 27.)

5C-13. Find the length of the curve $y = \ln \sin x$ for $\pi/4 \le x \le \pi/2$.

5C-14. Find the volume of one hump of $y = \sin ax$ revolved around the x-axis.

5D. Integration by inverse substitution

Evaluate the following integrals

5D-1. $\int \frac{dx}{(a^2-x^2)^{3/2}}$ 5D-2. $\int \frac{x^3 dx}{\sqrt{a^2-x^2}}$ 5D-3. $\int \frac{(x+1)dx}{4+x^2}$

5D-4. $\int \sqrt{a^2+x^2}dx$ 5D-5. $\int \frac{\sqrt{a^2-x^2}dx}{x^2}$ 5D-6. $\int x^2\sqrt{a^2+x^2}dx$

(For 5D-4,6 use $x = a\sinh y$, and $\cosh^2 y = (\cosh(2y)+1)/2$, $\sinh 2y = 2\sinh y \cosh y$.)

5D-7. $\int \frac{\sqrt{x^2-a^2}dx}{x^2}$ 5D-8. $\int x\sqrt{x^2-9}dx$

5D-9. Find the arclength of $y = \ln x$ for $1 \le x \le b$.

Completing the square

Calculate the following integrals

5D-10. $\displaystyle\int \frac{dx}{(x^2+4x+13)^{3/2}}$ 5D-11. $\displaystyle\int x\sqrt{-8+6x-x^2}\,dx$ 5D-12. $\displaystyle\int \sqrt{-8+6x-x^2}\,dx$

5D-13. $\displaystyle\int \frac{dx}{\sqrt{2x-x^2}}$ 5D-14. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+4x+13}}$ 5D-15. $\displaystyle\int \frac{\sqrt{4x^2-4x+17dx}}{2x-1}$

5E. Integration by partial fractions

5E-1. $\displaystyle\int \frac{dx}{(x-2)(x+3)}dx$ 5E-2. $\displaystyle\int \frac{x\,dx}{(x-2)(x+3)}dx$ 5E-3. $\displaystyle\int \frac{x\,dx}{(x^2-4)(x+3)}dx$

5E-4. $\displaystyle\int \frac{3x^2+4x-11}{(x^2-1)(x-2)}dx$ 5E-5. $\displaystyle\int \frac{3x+2}{x(x+1)^2}dx$ 5E-6. $\displaystyle\int \frac{2x-9}{(x^2+9)(x+2)}dx$

5E-7 The equality (1) of Notes F is valid for $x \neq 1, -2$. Therefore, the equality (4) is also valid only when $x \neq 1, -2$, since it arises from (1) by multiplication. Why then is it legitimate to substitute $x = 1$ into (4)?

5E-8 Express the following as a sum of a polynomial and a proper rational function

a) $\dfrac{x^2}{x^2-1}$ b) $\dfrac{x^3}{x^2-1}$ c) $\dfrac{x^2}{3x-1}$

d) $\dfrac{x+2}{3x-1}$ e) $\dfrac{x^8}{(x+2)^2(x-2)^2}$ (just give the form of the solution)

5E-9 Integrate the functions in Problem **5E-8**.

5E-10 Evaluate the following integrals

a) $\displaystyle\int \frac{dx}{x^3-x}$ b) $\displaystyle\int \frac{(x+1)dx}{(x-2)(x-3)}$ c) $\displaystyle\int \frac{(x^2+x+1)dx}{x^2+8x}$

d) $\displaystyle\int \frac{(x^2+x+1)dx}{x^2+8x}$ e) $\displaystyle\int \frac{dx}{x^3+x^2}$ f) $\displaystyle\int \frac{(x^2+1)dx}{x^3+2x^2+x}$

g) $\displaystyle\int \frac{x^3dx}{(x+1)^2(x-1)}$ h) $\displaystyle\int \frac{(x^2+1)dx}{x^2+2x+2}$

5E-11 Solve the differential equation $dy/dx = y(1-y)$.

5E-12 This problem shows how to integrate any rational function of $\sin\theta$ and $\cos\theta$ using the substitution $z = \tan(\theta/2)$. The integrand is transformed into a rational function of z, which can be integrated using the method of partial fractions.

a) Show that

$$\cos\theta = \frac{1-z^2}{1+z^2}, \quad \sin\theta = \frac{2z}{1+z^2}, \quad d\theta = \frac{2dz}{1+z^2}.$$

Calculate the following integrals using the substitution $z = \tan(\theta/2)$ of part (a).

b) $\displaystyle\int_0^{\pi} \frac{d\theta}{1+\sin\theta}$ c) $\displaystyle\int_0^{\pi} \frac{d\theta}{(1+\sin\theta)^2}$ d) $\displaystyle\int_0^{\pi} \sin\theta d\theta$ (Not the easiest way!)

5E-13 a) Use the polar coordinate formula for area to compute the area of the region $0 < r < 1/(1+\cos\theta)$, $0 \le \theta \le \pi/2$. Hint: Problem 12 shows how the substitution $z = \tan(\theta/2)$ allows you to integrate any rational function of a trigonometric function.

b) Compute this same area using rectangular coordinates and compare your answers.

5F. Integration by parts. Reduction formulas

Evaluate the following integrals

5F-1 a) $\displaystyle\int x^a \ln x dx$ $(a \ne -1)$ b) Evaluate the case $a = -1$ by substitution.

5F-2 a) $\displaystyle\int xe^x dx$ b) $\displaystyle\int x^2 e^x dx$ c) $\displaystyle\int x^3 e^x dx$

d) Derive the reduction formula expressing $\displaystyle\int x^n e^{ax} dx$ in terms of $\displaystyle\int x^{n-1} e^{ax} dx$.

5F-3 Evaluate $\displaystyle\int \sin^{-1}(4x) dx$

5F-4 Evaluate $\displaystyle\int e^x \cos x dx$. (Integrate by parts twice.)

5F-5 Evaluate $\displaystyle\int \cos(\ln x) dx$. (Integrate by parts twice.)

5F-6 Show the substitution $t = e^x$ transforms the integral $\displaystyle\int x^n e^x dx$, into $\displaystyle\int (\ln t)^n dt$. Use a reduction procedure to evaluate this integral.

Unit 6. Additional Topics

6A. Indeterminate forms; L'Hospital's rule

6A-1 Find the following limits

a) $\lim_{x\to 0} \frac{\sin 3x}{x}$ b) $\lim_{x\to 0} \frac{\cos(x/2)-1}{x^2}$ c) $\lim_{x\to\infty} \frac{\ln x}{x}$

d) $\lim_{x\to 0} \frac{x^2-3x-4}{x+1}$ e) $\lim_{x\to 0} \frac{\tan^{-1} x}{5x}$ f) $\lim_{x\to 0} \frac{x-\sin x}{x^3}$

g) $\lim_{x\to 1} \frac{x^a-1}{x^b-1}$ h) $\lim_{x\to 1} \frac{\tan(x)}{\sin(3x)}$ i) $\lim_{x\to\pi} \frac{\ln\sin(x/2)}{x-\pi}$

j) $\lim_{x\to\pi} \frac{\ln\sin(x/2)}{(x-\pi)^2}$

6A-2 Evaluate the following limits.

a) $\lim_{x\to 0+} x^x$ b) $\lim_{x\to 0+} x^{1/x}$ c) $\lim_{x\to 0+} (1/x)^{\ln x}$

d) $\lim_{x\to 0+} (\cos x)^{1/x}$ e) $\lim_{x\to\infty} x^{1/x}$ f) $\lim_{x\to 0+} (1+x^2)^{1/x}$

g) $\lim_{x\to 0+} (1+3x)^{10/x}$ h) $\lim_{x\to\infty} \frac{x+\cos x}{x}$ i) $\lim_{x\to\infty} x\sin\frac{1}{x}$

j) $\lim_{x\to 0+} \left(\frac{x}{\sin x}\right)^{1/x^2}$ k) $\lim_{x\to\infty} x^a(\ln x)^b$. Consider all values of a and b.

6A-3 The power x^{-1} is the exceptional case among the integrals of the powers of x. It would be nice if

$$\lim_{a\to -1} \int x^a dx = \int x^{-1} dx$$

It seems hopeless for this to be true[1] since

$$\int x^a dx = \frac{x^{a+1}}{a+1} + c \text{ for } a \neq -1$$

involves only powers, yet the integral of x^{-1} is a logarithm. But it can be rescued using the definite integral. Show using L'Hospital's rule that

$$\lim_{a\to -1} \int_1^x t^a dt = \int_1^x t^{-1} dt \quad (= \ln x)$$

6A-4 Show that as a tends to -1 of a well-chosen solution to E30/1(a) tends to the answer in part (b). Hint: Follow the method of the preceding problem.

6A-5 By repeated use of L'Hospital's rule,

$$\lim_{x\to 0} \frac{3x^2-4x}{2x-x^2} = \lim_{x\to 0} \frac{6x-4}{2-2x} = \lim_{x\to 0} \frac{6}{-2} = -3,$$

[1]It seems hopeless because for almost all choices of c the indefinite integral has an infinite limit as $a \to -1$. The definite integral leads to the correct choice of c, namely, $c = -1/(a+1)$. The constant c is a constant with respect to x, but there is no reason why it can't vary with a. And the right choice of c makes the limit as $a \to -1$ finite.

yet when $x \simeq 0$, $\dfrac{3x^2-4x}{2x-x^2} \simeq \dfrac{-4x}{2x} = -2$. Resolve the contradiction.

6A-6 Graph the following functions. (L'Hospital's rule will help with some of the limiting values at the ends.)

a) $y = xe^{-x}$ b) $y = x\ln x$ c) $y = x/\ln x$

6B. Improper integrals

Test the following improper integrals for convergence by using comparison with a simpler integral.

6B-1. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^3+5}}$ **6B-2.** $\displaystyle\int_0^\infty \frac{x^2dx}{x^3+2}$ **6B-3.** $\displaystyle\int_0^1 \frac{dx}{x^3+x^2}$

6B-4. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ **6B-5.** $\displaystyle\int_0^\infty \frac{e^{-x}dx}{x}$ **6B-6.** $\displaystyle\int_1^\infty \frac{\ln x\,dx}{x^2}$

6B-7 Decide whether the following integrals are convergent or divergent and evaluate if convergent.

a) $\displaystyle\int_0^\infty e^{-8x}dx$ b) $\displaystyle\int_1^\infty x^{-n}dx,\ n>1$ c) $\displaystyle\int_1^\infty x^{-n}dx,\ 0<n\le 1$

d) $\displaystyle\int_0^2 \frac{x\,dx}{\sqrt{4-x^2}}$ e) $\displaystyle\int_0^2 \frac{dx}{\sqrt{2-x}}$ f) $\displaystyle\int_e^\infty \frac{dx}{x(\ln x)^2}$

g) $\displaystyle\int_0^1 \frac{dx}{x^{1/3}}$ h) $\displaystyle\int_0^1 \frac{dx}{x^3}$ i) $\displaystyle\int_{-1}^1 \frac{dx}{x}$

j) $\displaystyle\int_0^1 \ln x\,dx$ k) $\displaystyle\int_0^\infty e^{-2x}\cos x\,dx$ l) $\displaystyle\int_e^\infty \frac{dx}{x(\ln x)}$. (Use (f).)

m) $\displaystyle\int_0^\infty \frac{dx}{(x+2)^3}$ n) $\displaystyle\int_0^\infty \frac{dx}{(x-2)^3}$ o) $\displaystyle\int_0^{10} \frac{(\ln x)^2}{x}dx$

p) $\displaystyle\int_0^\pi \sec x\,dx$

6B-8 Find the following limits. (Use the fundamental theorem of calculus.)

a) $\displaystyle\lim_{x\to\infty} e^{-x^2}\int_0^x e^{t^2}dt$ b) $\displaystyle\lim_{x\to\infty} xe^{-x^2}\int_0^x e^{t^2}dt$ c) $\displaystyle\lim_{x\to\infty} e^{x^2}\int_0^x e^{-t^2}dt$

d) $\displaystyle\lim_{a\to 0+} \sqrt{a}\int_a^1 \frac{dx}{\sqrt{x}}$ e) $\displaystyle\lim_{a\to 0+} \sqrt{a}\int_a^1 \frac{dx}{x^{3/2}}$ f) $\displaystyle\lim_{b\to(\pi/2)^+} (b-\pi/2)\int_0^b \frac{dx}{1-\sin x}$

6C. Infinite Series

6C-1 Find the sum of the following geometric series:

a) $1+1/5+1/25+\cdots$ b) $8+2+1/2+\cdots$ c) $1/4+1/5+\cdots$

Write the two following infinite decimals as the quotient of two integers:

d) $0.4444\ldots$ e) $0.0602602602602\ldots$

6C-2 Decide whether the following series are convergent or divergent; indicate reasoning. (Do not evaluate the sum.)

a) $1+1/2+1/3+1/4+1/5+\cdots$; use comparison with an integral.

b) $\sum_{n=1}^{\infty}\frac{1}{n^p}$; consider the cases $p>1$ and $p\le 1$.

c) $1/2+1/4+1/6+1/8+\cdots$

d) $1+1/3+1/5+1/7+\cdots$

e) $1-1/2+1/3-1/4+1/5-\cdots$ Hint: Combine pairs of consecutive terms to take advantage of the cancellation. Then use comparison.

f) $\sum_{n=1}^{\infty}\frac{n}{n!}$. g) $\sum_{n=1}^{\infty}\left(\frac{\sqrt{5}-1}{2}\right)^n$. h) $\sum_{n=1}^{\infty}\left(\frac{\sqrt{5}+1}{2}\right)^n 5^{-n/2}$.

i) $\sum_{n=1}^{\infty}\frac{\ln n}{n}$. j) $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$. k) $\sum_{n=1}^{\infty}\frac{n+2}{n^4-5}$.

l) $\sum_{n=1}^{\infty}\frac{(n+2)^{1/3}}{(n^4+5)^{1/3}}$. m) $\sum_{n=1}^{\infty}\ln(\cos\frac{1}{n})$ n) $\sum_{n=1}^{\infty}n^2e^{-n}$

o) $\sum_{n=1}^{\infty}n^2e^{-\sqrt{n}}$

6C-3 a) Use the upper and lower Riemann sums of

$$\ln n=\int_1^n\frac{dx}{x}$$

to show that

$$\ln n<1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}<1+\ln n$$

b) Suppose that it takes 10^{-10} seconds for a computer to add one term in the series $\sum 1/n$. About how long would it take for the partial sum to reach 1000?

7. Infinite Series

7A. Basic Definitions

7A-1 Do the following series converge or diverge? Give reason. If the series converges, find its sum.

a) $1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots+\frac{1}{4^n}+\ldots$

b) $1-1+1-1+\ldots+(-1)^n+\ldots$

c) $1+\frac{1}{2}+\frac{2}{3}+\ldots+\frac{n}{n+1}+\ldots$

d) $\ln 2+\ln\sqrt{2}+\ln\sqrt[3]{2}+\ln\sqrt[4]{2}+\ldots$

e) $\sum_1^\infty \frac{2^{n-1}}{3^n}$

f) $\sum_0^\infty (-1)^n \frac{1}{3^n}$

7A-2 Find the rational number represented by the infinite decimal .21111... .

7A-3 For which x does the series $\sum_0^\infty \left(\frac{x}{2}\right)^n$ converge? For these values, find its sum $f(x)$.

7A-4 Find the sum of these series by first finding the partial sum S_n.

a) $\sum_1^\infty \left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$

b) $\sum_1^\infty \frac{1}{n(n+2)}$. (Hint: $\frac{1}{n(n+2)}=\frac{a}{n}+\frac{b}{n+2}$ for suitable a, b).

7A-5 A ball is dropped from height h; each time it lands, it bounces back 2/3 of the height from which it previously fell. What is the total distance (up and down) the ball travels?

7B: Convergence Tests

7B-1 Using the integral test, tell whether the following series converge or diverge; show work or reasoning.

a) $\sum_0^\infty \frac{n}{n^2+4}$ b) $\sum_0^\infty \frac{1}{n^2+1}$ c) $\sum_0^\infty \frac{1}{\sqrt{n+1}}$

d) $\sum_1^\infty \frac{\ln n}{n}$ e) $\sum_2^\infty \frac{1}{(\ln n)^p \cdot n}$ f) $\sum_1^\infty \frac{1}{n^p}$

(In the last two, the answer depends on the value of the parameter p.)

7B-2 Using the limit comparison test, tell whether each series converges or diverges; show work or reasoning. (For some of them, simple comparison works.)

a) $\sum_1^\infty \frac{1}{n^2+3n}$ b) $\sum_1^\infty \frac{1}{n+\sqrt{n}}$ c) $\sum_1^\infty \frac{1}{\sqrt{n^2+n}}$

d) $\sum_1^\infty \sin\left(\frac{1}{n^2}\right)$ e) $\sum_1^\infty \frac{\sqrt{n}}{n^2+1}$ f) $\sum_1^\infty \frac{\ln n}{n}$

g) $\sum_2^\infty \frac{n^2}{n^4-1}$ h) $\sum_1^\infty \frac{n^3}{4n^4+n^2}$

7B-3 Prove that if $a_n > 0$ and $\sum_0^\infty a_n$ converges, then $\sum_0^\infty \sin a_n$ also converges.

7B-4 Using the ratio test, or otherwise, determine whether or not each of these series is absolutely convergent. (Note that $0! = 1$.)

a) $\displaystyle\sum_0^\infty \frac{n}{2^n}$ b) $\displaystyle\sum_0^\infty \frac{2^n}{n!}$ c) $\displaystyle\sum_1^\infty \frac{2^n}{1\cdot 3\cdot 5\cdots(2n-1)}$

d) $\displaystyle\sum_0^\infty \frac{(n!)^2}{(2n)!}$ e) $\displaystyle\sum_1^\infty \frac{(-1)^n}{\sqrt{n}}$ f) $\displaystyle\sum_1^\infty \frac{n!}{n^n}$; use $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

g) $\displaystyle\sum_1^\infty \frac{(-1)^n}{n^2}$ h) $\displaystyle\sum_0^\infty \frac{(-1)^n}{\sqrt{n^2+1}}$ i) $\displaystyle\sum_0^\infty \frac{n}{n+1}$

7B-5 For those series in **7C-4** which are *not* absolutely convergent, tell whether they are conditionally convergent or divergent.

7B-6 By using the ratio test, determine the radius of convergence of each of the following power series.

a) $\displaystyle\sum_1^\infty \frac{x^n}{n}$ b) $\displaystyle\sum_1^\infty \frac{2^n x^n}{n^2}$ c) $\displaystyle\sum_0^\infty n!x^n$

d) $\displaystyle\sum_0^\infty \frac{(-1)^n x^{2n}}{3^n}$ e) $\displaystyle\sum_0^\infty \frac{(-1)^n x^{2n+1}}{2^n\sqrt{n}}$ f) $\displaystyle\sum_0^\infty \frac{(2n)!x^{2n}}{(n!)^2}$

g) $\displaystyle\sum_2^\infty \frac{x^n}{\ln n}$ h) $\displaystyle\sum_0^\infty \frac{2^{2n}x^n}{n!}$

7C: Taylor Approximations and Power Series

7C-1 Using the general formula for the coefficients a_n, find the Taylor series at 0 for the following functions; do the work systematically, calculating in order the $f^{(n)}, f^{(n)}(0)$, and then the a_n.

a) $\cos x$ b) $\ln(1+x)$ c) $\sqrt{1+x}$

7C-2 Calculate $\sin 1$ using the Taylor series up to the term in x^3. Estimate the accuracy using the remainder term. (The calculator value is .84147.) Use the remainder term $R_6(x)$, not $R_5(x)$; why?

7C-3 Using the remainder term, tell for what value of n in the approximation

$$e^x \approx 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$$

the resulting calculation will give e to 3 decimal places (by convention, this means: within .0005).

7C-4 By using the remainder term, tell whether $\cos x \approx 1 - \dfrac{x^2}{2!}$ will be valid to within .001 over the interval $|x| < .5$.

7C-5 Calculate $\displaystyle\int_0^{.5} e^{-x^2}\,dx$, using the approximation for e^{-x^2} up to the term in x^4. Estimate the error, using the correct remainder term (cf. **7B-3**), and tell whether the answer will be good to 3 decimal places.

7D: General Power Series

7D-1 Find the power series around $x = 0$ for each of the following functions by using known Taylor series: use substitution, addition, differentiation, integration, or anything else you can think of:

a) e^{-2x} b) $\cos\sqrt{x},\ x \geq 0$ c) $\sin^2 x$ (use an identity)

d) $\dfrac{1}{(1+x)^2}$ e) $\tan^{-1} x$ (differentiate) f) $\ln(1+x)$

g) $\cosh x = \dfrac{e^x + e^{-x}}{2}$

7D-2 By using operations on power series (substitution, addition, integration, differentiation, multiplication), find the power series for the following functions, and determine the radius of convergence. (Where indicated, give just the first 2 or 3 non-zero terms.)

a) $\dfrac{1}{x+9}$ b) e^{-x^2} c) $e^x \cos x$ (3 terms)

d) $\displaystyle\int_0^x \frac{\sin t}{t}\,dt$ e) $\operatorname{erf} x = \displaystyle\int_0^x e^{-t^2/2}\,dt$ f) $\dfrac{1}{x^3-1}$

g) $\cos^2 x$ (differentiate; then use a trigonometric identity)

h) $\dfrac{\sin x}{1-x}$ (3 terms); do it two ways: multiplication, and dividing $\sin x$ series by $1-x$

i) $\tan x$ (2 terms); do it two ways: Taylor series, and division of power series

7D-3 Find the following limits by using power series, *not* by using L'Hospital's rule.

a) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2}$ b) $\displaystyle\lim_{x\to 0} \frac{x-\sin x}{x^3}$

c) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x}-1-x/2}{\sin^2 x}$ d) $\displaystyle\lim_{u\to 0} \frac{\cos u - 1}{\ln(1+u) - u}$