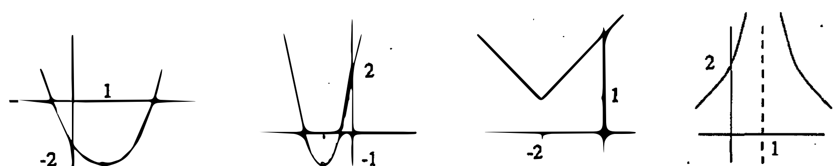
Solutions of General Mathematics Exercises

Unit 1. Differentiation

1A. Graphing

1A-1,2 a) $y = (x-1)^2 - 2$

b) $y = 3(x^2 + 2x) + 2 = 3(x+1)^2 - 1$



1a 1b 2a 2b

1A-3 a) $f(-x) = \dfrac{(-x)^3 - 3x}{1-(-x)^4} = \dfrac{-x^3 - 3x}{1-x^4} = -f(x)$, so it is odd.

b) $(\sin(-x))^2 = (\sin x)^2$, so it is even.

c) $\dfrac{\text{odd}}{\text{even}}$, so it is odd

d) $(1-x)^4 \neq \pm(1+x)^4$: neither.

e) $J_0((-x)^2) = J_0(x^2)$, so it is even.

1A-4 a) $p(x) = p_e(x) + p_o(x)$, where $p_e(x)$ is the sum of the even powers and $p_o(x)$ is the sum of the odd powers

b) $f(x) = \dfrac{f(x)+f(-x)}{2} + \dfrac{f(x)-f(-x)}{2}$

$F(x) = \dfrac{f(x)+f(-x)}{2}$ is even and $G(x) = \dfrac{f(x)-f(-x)}{2}$ is odd because

$$F(-x) = \frac{f(-x)+f(-(-x))}{2} = F(x); \qquad G(-x) = \frac{f(x)-f(-x)}{2} = -G(-x).$$

c) Use part b:

$$\frac{1}{x+a} + \frac{1}{-x+a} = \frac{2a}{(x+a)(-x+a)} = \frac{2a}{a^2-x^2} \quad \text{even}$$

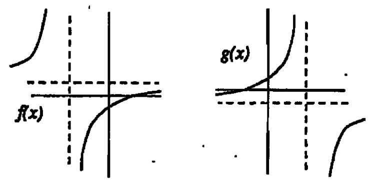
$$\frac{1}{x+a} - \frac{1}{-x+a} = \frac{-2x}{(x+a)(-x+a)} = \frac{-2x}{a^2-x^2} \quad \text{odd}$$

$$\Longrightarrow \frac{1}{x+a} = \frac{a}{a^2-x^2} - \frac{x}{a^2-x^2}$$

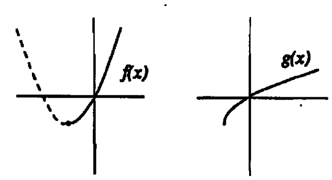
1A-5 a) $y = \dfrac{x-1}{2x+3}$. Crossmultiply and solve for x, getting $x = \dfrac{3y+1}{1-2y}$, so the inverse function is $\dfrac{3x+1}{1-2x}$.

b) $y = x^2 + 2x = (x+1)^2 - 1$

(Restrict domain to $x \le -1$, so when it's flipped about the diagonal $y = x$, you'll still get the graph of a function.) Solving for x, we get $x = \sqrt{y+1} - 1$, so the inverse function is $y = \sqrt{x+1} - 1$.



5a



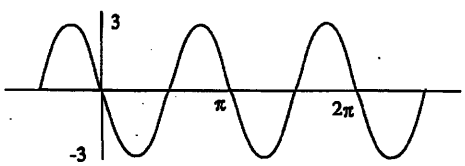
5b

1A-6 a) $A = \sqrt{1+3} = 2$, $\tan c = \dfrac{\sqrt{3}}{1}$, $c = \frac{\pi}{3}$. So $\sin x + \sqrt{3}\cos x = 2\sin(x + \frac{\pi}{3})$.

b) $\sqrt{2}\sin(x - \dfrac{\pi}{4})$

1A-7 a) $3\sin(2x - \pi) = 3\sin 2(x - \dfrac{\pi}{2})$, amplitude 3, period π, phase angle $\pi/2$.

b) $-4\cos(x + \dfrac{\pi}{2}) = 4\sin x$ amplitude 4, period 2π, phase angle 0.



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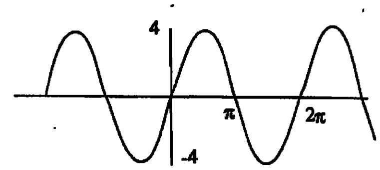


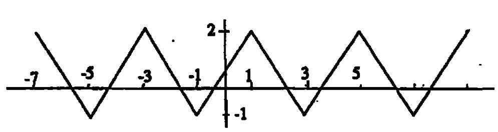
7b

1A-8

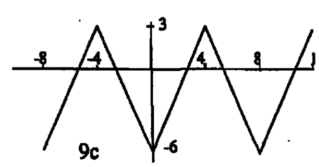
$f(x)$ odd $\Longrightarrow f(0) = -f(0) \Longrightarrow f(0) = 0$.

So $f(c) = f(2c) = \cdots = 0$, also (by periodicity, where c is the period).

1A-9



9ab period = 4



9c

c) The graph is made up of segments joining $(0, -6)$ to $(4, 3)$ to $(8, -6)$. It repeats in a zigzag with period 8. * This can be derived using:

$$x/2 - 1 = -1 \implies x = 0 \text{ and } g(0) = 3f(-1) - 3 = -6$$
$$x/2 - 1 = 1 \implies x = 4 \text{ and } g(4) = 3f(1) - 3 = 3$$
$$x/2 - 1 = 3 \implies x = 8 \text{ and } g(8) = 3f(3) - 3 = -6$$

1B. Velocity and rates of change

1B-1 a) $h =$ height of tube $= 400 - 16t^2$.

$$\text{average speed } \frac{h(2)-h(0)}{2} = \frac{(400-16\cdot 2^2)-400}{2} = -32\text{ft/sec}$$

(The minus sign means the test tube is going down. You can also do this whole problem using the function $s(t) = 16t^2$, representing the distance down measured from the top. Then all the speeds are positive instead of negative.)

b) Solve $h(t) = 0$ (or $s(t) = 400$) to find landing time $t = 5$. Hence the average speed for the last two seconds is

$$\frac{h(5)-h(3)}{2} = \frac{0-(400-16\cdot 3^2)}{2} = -128\text{ft/sec}$$

c)

$$\frac{h(t)-h(5)}{t-5} = \frac{400-16t^2-0}{t-5} = \frac{16(5-t)(5+t)}{t-5}$$
$$= -16(5+t) \to -160\text{ft/sec as } t \to 5$$

1B-2 A tennis ball bounces so that its initial speed straight upwards is b feet per second. Its height s in feet at time t seconds is

$$s = bt - 16t^2$$

a)

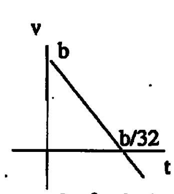
$$\begin{aligned}\frac{s(t+h)-s(t)}{h} &= \frac{b(t+h)-16(t+h)^2-(bt-16t^2)}{h}\\ &= \frac{bt+bh-16t^2-32th-16h^2-bt+16t^2}{h}\\ &= \frac{bh-32th-16h^2}{h}\\ &= b-32t-16h \to b-32t \text{ as } h\to 0\end{aligned}$$

Therefore, $v = b - 32t$.

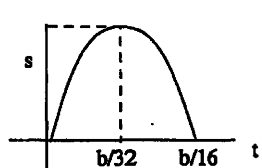
b) The ball reaches its maximum height exactly when the ball has finished going up. This is time at which $v(t) = 0$, namely, $t = b/32$.

c) The maximum height is $s(b/32) = b^2/64$.

d) The graph of v is a straight line with slope -32. The graph of s is a parabola with maximum at place where $v = 0$ at $t = b/32$ and landing time at $t = b/16$.



graph of velocity



graph of position

e) If the initial velocity on the first bounce was $b_1 = b$, and the velocity of the second bounce is b_2, then $b_2^2/64 = (1/2)b_1^2/64$. Therefore, $b_2 = b_1/\sqrt{2}$. The second bounce is at $b_1/16 + b_2/16$.

(continued →)

f) If the ball continues to bounce then the landing times form a geometric series

$$\begin{aligned} b_1/16 + b_2/16 + b_3/16 + \cdots &= b/16 + b/16\sqrt{2} + b/16(\sqrt{2})^2 + \cdots \\ &= (b/16)(1 + (1/\sqrt{2}) + (1/\sqrt{2})^2 + \cdots) \\ &= \frac{b/16}{1 - (1/\sqrt{2})} \end{aligned}$$

Put another way, the ball stops bouncing after $1/(1-(1/\sqrt{2})) \approx 3.4$ times the length of time the first bounce.

1C. Slope and derivative.

1C-1 a)

$$\begin{aligned} \frac{\pi(r+h)^2 - \pi r^2}{h} &= \frac{\pi(r^2+2rh+h^2) - \pi r^2}{h} = \frac{\pi(2rh+h^2)}{h} \\ &= \pi(2r+h) \\ &\to 2\pi r \ \text{ as } h \to 0 \end{aligned}$$

b)

$$\begin{aligned} \frac{(4\pi/3)(r+h)^3 - (4\pi/3)r^3}{h} &= \frac{(4\pi/3)(r^3+3r^2h+3rh^2+h^3) - (4\pi/3)r^3}{h} \\ &= \frac{(4\pi/3)(3r^2h+3rh^2+h^3)}{h} \\ &= (4\pi/3)(3r^2+3rh+h^2) \\ &\to 4\pi r^2 \ \text{ as } h \to 0 \end{aligned}$$

1C-2 $\dfrac{f(x)-f(a)}{x-a} = \dfrac{(x-a)g(x)-0}{x-a} = g(x) \to g(a)$ as $x \to a$.

1C-3 a)

$$\begin{aligned} \frac{1}{h}\left[\frac{1}{2(x+h)+1} - \frac{1}{2x+1}\right] &= \frac{1}{h}\left[\frac{2x+1-(2(x+h)+1)}{(2(x+h)+1)(2x+1)}\right] \\ &= \frac{1}{h}\left[\frac{-2h}{(2(x+h)+1)(2x+1)}\right] \\ &= \frac{-2}{(2(x+h)+1)(2x+1)} \\ &\longrightarrow \frac{-2}{(2x+1)^2} \ \text{ as } h \to 0 \end{aligned}$$

b)

$$\begin{aligned}\frac{2(x+h)^2+5(x+h)+4-(2x^2+5x+4)}{h} &= \frac{2x^2+4xh+2h^2+5x+5h-2x^2-5x}{h}\\ &= \frac{4xh+2h^2+5h}{h} = 4x+2h+5\\ &\longrightarrow 4x+5 \text{ as } h\to 0\end{aligned}$$

c)

$$\begin{aligned}\frac{1}{h}\left[\frac{1}{(x+h)^2+1}-\frac{1}{x^2+1}\right] &= \frac{1}{h}\left[\frac{(x^2+1)-((x+h)^2+1)}{((x+h)^2+1)(x^2+1)}\right]\\ &= \frac{1}{h}\left[\frac{x^2+1-x^2-2xh-h^2-1}{((x+h)^2+1)(x^2+1)}\right]\\ &= \frac{1}{h}\left[\frac{-2xh-h^2}{((x+h)^2+1)(x^2+1)}\right]\\ &= \frac{-2x-h}{((x+h)^2+1)(x^2+1)}\\ &\longrightarrow \frac{-2x}{(x^2+1)^2} \text{ as } h\to 0\end{aligned}$$

d) Common denominator:

$$\frac{1}{h}\left[\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}\right] = \frac{1}{h}\left[\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}\right]$$

Now simplify the numerator by multiplying numerator and denominator by $\sqrt{x}+\sqrt{x+h}$, and using $(a-b)(a+b)=a^2-b^2$:

$$\begin{aligned}\frac{1}{h}\left[\frac{(\sqrt{x})^2-(\sqrt{x+h})^2}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}\right] &= \frac{1}{h}\left[\frac{x-(x+h)}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}\right]\\ &= \frac{1}{h}\left[\frac{-h}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}\right]\\ &= \left[\frac{-1}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}\right]\\ &\longrightarrow \frac{-1}{2(\sqrt{x})^3} = -\frac{1}{2}x^{-3/2} \text{ as } h\to 0\end{aligned}$$

e) For part (a), $-2/(2x+1)^2<0$, so there are no points where the slope is 1 or 0. For slope -1,

$$-2/(2x+1)^2=-1 \implies (2x+1)^2=2 \implies 2x+1=\pm\sqrt{2} \implies x=-1/2\pm\sqrt{2}/2$$

For part (b), the slope is 0 at $x=-5/4$, 1 at $x=-1$ and -1 at $x=-3/2$.

1C-4 Using Problem 3,

a) $f'(1) = -2/9$ and $f(1) = 1/3$, so $y = -(2/9)(x-1) + 1/3 = (-2x+5)/9$

b) $f(a) = 2a^2 + 5a + 4$ and $f'(a) = 4a + 5$, so

$$y = (4a+5)(x-a) + 2a^2 + 5a + 4 = (4a+5)x - 2a^2 + 4$$

c) $f(0) = 1$ and $f'(0) = 0$, so $y = 0(x-0) + 1$, or $y = 1$.

d) $f(a) = 1/\sqrt{a}$ and $f'(a) = -(1/2)a^{-3/2}$, so

$$y = -(1/2)a^{3/2}(x-a) + 1/\sqrt{a} = -a^{-3/2}x + (3/2)a^{-1/2}$$

1C-5 Method 1. $y'(x) = 2(x-1)$, so the tangent line through $(a, 1+(a-1)^2)$ is

$$y = 2(a-1)(x-a) + 1 + (a-1)^2$$

In order to see if the origin is on this line, plug in $x = 0$ and $y = 0$, to get the following equation for a.

$$0 = 2(a-1)(-a) + 1 + (a-1)^2 = -2a^2 + 2a + 1 + a^2 - 2a + 1 = -a^2 + 2$$

Therefore $a = \pm\sqrt{2}$ and the two tangent lines through the origin are

$$y = 2(\sqrt{2}-1)x \text{ and } y = -2(\sqrt{2}+1)x$$

(Because these are lines throught the origin, the constant terms must cancel: this is a good check of your algebra!)

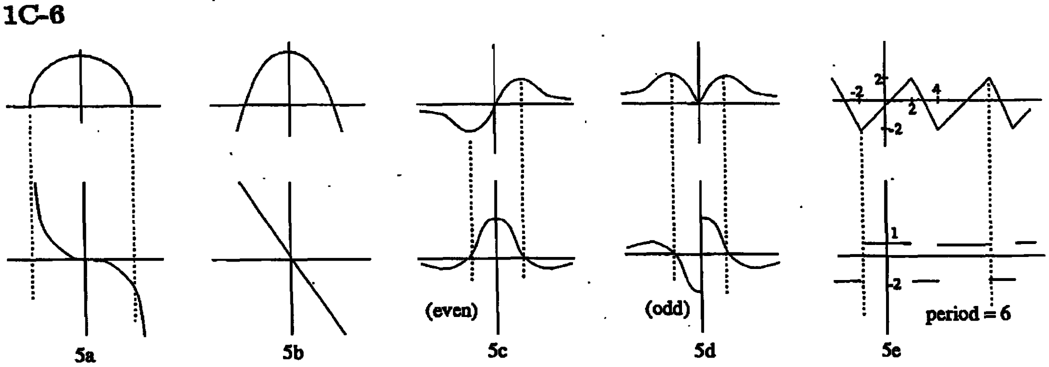
Method 2. Seek tangent lines of the form $y = mx$. Suppose that $y = mx$ meets $y = 1 + (x-1)^2$, at $x = a$, then $ma = 1 + (a-1)^2$. In addition we want the slope $y'(a) = 2(a-1)$ to be equal to m, so $m = 2(a-1)$. Substituting for m we find

$$2(a-1)a = 1 + (a-1)^2$$

This is the same equation as in method 1: $a^2 - 2 = 0$, so $a = \pm\sqrt{2}$ and $m = 2(\pm\sqrt{2} - 1)$, and the two tangent lines through the origin are as above,

$$y = 2(\sqrt{2}-1)x \text{ and } y = -2(\sqrt{2}+1)x$$

1C-6



1D. Limits and continuity

1D-1 Calculate the following limits if they exist. If they do not exist, then indicate whether they are $+\infty$, $-\infty$ or undefined.

a) -4

b) $8/3$

c) undefined (both $\pm\infty$ are possible)

d) Note that $2-x$ is negative when $x>2$, so the limit is $-\infty$

e) Note that $2-x$ is positive when $x<2$, so the limit is $+\infty$ (can also be written ∞)

f) $\dfrac{4x^2}{x-2} = \dfrac{4x}{1-(2/x)} \to \dfrac{\infty}{1} = \infty$ as $x \to \infty$

g) $\dfrac{4x^2}{x-2} - 4x = \dfrac{4x^2-4x(x-2)}{x-2} = \dfrac{8x}{x-2} = \dfrac{8}{1-(2/x)} \to 8$ as $x\to\infty$

i) $\dfrac{x^2+2x+3}{3x^2-2x+4} = \dfrac{1+(2/x)+(3/x^2)}{3-(2/x)+4/x^2)} \to \dfrac{1}{3}$ as $x\to\infty$

j) $\dfrac{x-2}{x^2-4} = \dfrac{x-2}{(x-2)(x+2)} = \dfrac{1}{x+2} \to \dfrac{1}{4}$ as $x \to 2$

1D-2 a) $\lim\limits_{x\to 0}\sqrt{x} = 0$ b) $\lim\limits_{x\to 1+}\dfrac{1}{x-1} = \infty$ $\lim\limits_{x\to 1-}\dfrac{1}{x-1} = -\infty$

c) $\lim\limits_{x\to 1}(x-1)^{-4} = \infty$ (left and right hand limits are same)

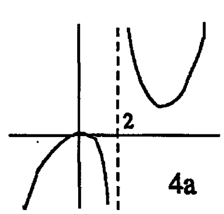
d) $\lim\limits_{x\to 0}|\sin x| = 0$ (left and right hand limits are same)

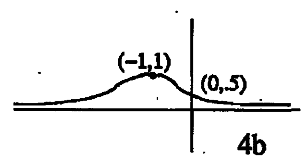
e) $\lim\limits_{x\to 0+}\dfrac{|x|}{x} = 1$ $\lim\limits_{x\to 0-}\dfrac{|x|}{x} = -1$

1D-3 a) $x=2$ removable $x=-2$ infinite b) $x=0,\pm\pi,\pm 2\pi, \ldots$ infinite

c) $x=0$ removable d) $x=0$ removable e) $x=0$ jump f) $x=0$ removable

1D-4





1D-5 a) for continuity, want $ax+b=1$ when $x=1$. Ans.: all a,b such that $a+b=1$

b) $\dfrac{dy}{dx} = \dfrac{d(x^2)}{dx} = 2x = 2$ when $x=1$. We have also $\dfrac{d(ax+b)}{dx} = a$. Therefore, to make $f'(x)$ continuous, we want $a=2$.

Combining this with the condition $a+b$=1 from part (a), we get finally $b=-1$, $a=2$.

1D-6 a) $f(0) = 0^2 + 4 \cdot 0 + 1 = 1$. Match the function values:

$$f(0^-) = \lim_{x \to 0} ax + b = b, \quad \text{so } b = 1 \text{ by continuity.}$$

Next match the slopes:

$$f'(0^+) = \lim_{x \to 0} 2x + 4 = 4$$

and $f'(0^-) = a$. Therefore, $a = 4$, since $f'(0)$ exists.

b)

$$f(1) = 1^2 + 4 \cdot 1 + 1 = 6 \text{ and } f(1^-) = \lim_{x \to 1} ax + b = a + b$$

Therefore continuity implies $a + b = 6$. The slope from the right is

$$f'(1^+) = \lim_{x \to 1} 2x + 4 = 6$$

Therefore, this must equal the slope from the left, which is a. Thus, $a = 6$ and $b = 0$.

1D-7

$$f(1) = c1^2 + 4 \cdot 1 + 1 = c + 5 \text{ and } f(1^-) = \lim_{x \to 1} ax + b = a + b$$

Therefore, by continuity, $c + 5 = a + b$. Next, match the slopes from left and right:

$$f'(1^+) = \lim_{x \to 1} 2cx + 4 = 2c + 4 \text{ and } f'(1^-) = \lim_{x \to 1} a = a$$

Therefore,

$$a = 2c + 4 \text{ and } b = -c + 1.$$

1D-8

a)

$$f(0) = \sin(2 \cdot 0) = 0 \text{ and } f(0^+) = \lim_{x \to 0} ax + b = b$$

Therefore, continuity implies $b = 0$. The slope from each side is

$$f'(0^-) = \lim_{x \to 0} 2\cos(2x) = 2 \text{ and } f'(0^+) = \lim_{x \to 0} a = a$$

Therefore, we need $a \neq 2$ in order that f not be differentiable.

b)

$$f(0) = \cos(2 \cdot 0) = 1 \text{ and } f(0^+) = \lim_{x \to 0} ax + b = b$$

Therefore, continuity implies $b = 1$. The slope from each side is

$$f'(0^-) = \lim_{x \to 0} -2\sin(2x) = 0 \text{ and } f'(0^+) = \lim_{x \to 0} a = a$$

Therefore, we need $a \neq 0$ in order that f not be differentiable.

1D-9 There cannot be any such values because every differentiable function is continuous.

1E: Differentiation formulas: polynomials, products, quotients

1E-1 Find the derivative of the following polynomials

a) $10x^9 + 15x^4 + 6x^2$

b) 0 ($e^2 + 1 \approx 8.4$ is a constant and the derivative of a constant is zero.)

c) $1/2$

d) By the product rule: $(3x^2+1)(x^5+x^2)+(x^3+x)(5x^4+2x) = 8x^7+6x^5+5x^4+3x^2$. Alternatively, multiply out the polynomial first to get $x^8+x^6+x^5+x^3$ and then differentiate.

1E-2 Find the antiderivative of the following polynomials

a) $ax^2/2 + bx + c$, where a and b are the given constants and c is a third constant.

b) $x^7/7 + (5/6)x^6 + x^4 + c$

c) The only way to get at this is to multiply it out: $x^6 + 2x^3 + 1$. Now you can take the antiderivative of each separate term to get

$$\frac{x^7}{7} + \frac{x^4}{2} + x + c$$

Warning: The answer is not $(1/3)(x^3+1)^3$. (The derivative does not match if you apply the chain rule, the rule to be treated below in E4.)

1E-3 $y' = 3x^2 + 2x - 1 = 0 \implies (3x-1)(x+1) = 0$. Hence $x = 1/3$ or $x = -1$ and the points are $(1/3, 49/27)$ and $(-1, 3)$

1E-4 a) $f(0) = 4$, and $f(0^-) = \lim_{x\to 0} 5x^5 + 3x^4 + 7x^2 + 8x + 4 = 4$. Therefore the function is continuous for all values of the parameters.

$$f'(0^+) = \lim_{x\to 0} 2ax + b = b \text{ and } f'(0^-) = \lim_{x\to 0} 25x^4 + 12x^3 + 14x + 8 = 8$$

Therefore, $b = 8$ and a can have any value.

b) $f(1) = a + b + 4$ and $f(1^+) = 5 + 3 + 7 + 8 + 4 = 27$. So by continuity,

$$a + b = 23$$

$$f'(1^-) = \lim_{x\to 1} 2ax + b = 2a + b; \qquad f'(1^+) = \lim_{x\to 1} 25x^4 + 12x^3 + 14x + 8 = 59.$$

Therefore, differentiability implies

$$2a + b = 59$$

Subtracting the first equation, $a = 59 - 23 = 36$ and hence $b = -13$.

1E-5 a) $\dfrac{1}{(1+x)^2}$ b) $\dfrac{1-2ax-x^2}{(x^2+1)^2}$ c) $\dfrac{-x^2-4x-1}{(x^2-1)^2}$

d) $3x^2 - 1/x^2$

1F. Chain rule, implicit differentiation

1F-1 a) Let $u = (x^2 + 2)$

$$\frac{d}{dx}u^2 = \frac{du}{dx}\frac{d}{du}u^2 = (2x)(2u) = 4x(x^2+2) = 4x^3 + 8x$$

Alternatively,

$$\frac{d}{dx}(x^2+2)^2 = \frac{d}{dx}(x^4 + 4x^2 + 4) = 4x^3 + 8x$$

b) Let $u = (x^2 + 2)$; then $\frac{d}{dx}u^{100} = \frac{du}{dx}\frac{d}{du}u^{100} = (2x)(100u^{99}) = (200x)(x^2+2)^{99}$.

1F-2 Product rule and chain rule:

$$10x^9(x^2+1)^{10} + x^{10}[10(x^2+1)^9(2x)] = 10(3x^2+1)x^9(x^2+1)^9$$

1F-3 $y = x^{1/n} \implies y^n = x \implies ny^{n-1}y' = 1$. Therefore,

$$y' = \frac{1}{ny^{n-1}} = \frac{1}{n}y^{1-n} = \frac{1}{n}x^{\frac{1}{n}-1}$$

1F-4 $(1/3)x^{-2/3} + (1/3)y^{-2/3}y' = 0$ implies

$$y' = -x^{-2/3}y^{2/3}$$

Put $u = 1 - x^{1/3}$. Then $y = u^3$, and the chain rule implies

$$\frac{dy}{dx} = 3u^2\frac{du}{dx} = 3(1-x^{1/3})^2(-(1/3)x^{-2/3}) = -x^{-2/3}(1-x^{1/3})^2$$

The chain rule answer is the same as the one using implicit differentiation because

$$y = (1-x^{1/3})^3 \implies y^{2/3} = (1-x^{1/3})^2$$

1F-5 Implicit differentiation gives $\cos x + y'\cos y = 0$. Horizontal slope means $y' = 0$, so that $\cos x = 0$. These are the points $x = \pi/2 + k\pi$ for every integer k. Recall that $\sin(\pi/2 + k\pi) = (-1)^k$, i.e., 1 if k is even and -1 if k is odd. Thus at $x = \pi/2 + k\pi$, $\pm 1 + \sin y = 1/2$, or $\sin y = \mp 1 + 1/2$. But $\sin y = 3/2$ has no solution, so the only solutions are when k is even and in that case $\sin y = -1 + 1/2$, so that $y = -\pi/6 + 2n\pi$ or $y = 7\pi/6 + 2n\pi$. In all there are two grids of points at the vertices of squares of side 2π, namely the points

$$(\pi/2 + 2k\pi, -\pi/6 + 2n\pi) \text{ and } (\pi/2 + 2k\pi, 7\pi/6 + 2n\pi); \qquad k, n \text{ any integers.}$$

1F-6 Following the hint, let $z = -x$. If f is even, then $f(x) = f(z)$ Differentiating and using the chain rule:

$$f'(x) = f'(z)(dz/dx) = -f'(z) \quad \text{because } dz/dx = -1$$

But this means that f' is odd. Similarly, if g is odd, then $g(x = -g(z)$. Differentiating and using the chain rule:

$$g'(x) = -g'(z)(dz/dx) = g'(z) \quad \text{because } dz/dx = -1$$

1F-7 a) $\dfrac{dD}{dx} = \dfrac{1}{2}((x-a)^2 + {y_0}^2)^{-1/2}(2(x-a)) = \dfrac{x-a}{\sqrt{(x-a)^2+{y_0}^2}}$

b) $\dfrac{dm}{dv} = m_0 \cdot \dfrac{-1}{2}(1 - v^2/c^2)^{-3/2} \cdot \dfrac{-2v}{c^2} = \dfrac{m_0 v}{c^2(1-v^2/c^2)^{3/2}}$

c) $\dfrac{dF}{dr} = mg \cdot (-\dfrac{3}{2})(1+r^2)^{-5/2} \cdot 2r = \dfrac{-3mgr}{(1+r^2)^{5/2}}$

d) $\dfrac{dQ}{dt} = at \cdot \dfrac{-6bt}{(1+bt^2)^4} + \dfrac{a}{(1+bt^2)^3} = \dfrac{a(1-5bt^2)}{(1+bt^2)^4}$

1F-8 a) $V = \dfrac{1}{3}\pi r^2 h \implies 0 = \dfrac{1}{3}\pi(2rr'h + r^2) \implies r' = \dfrac{-r^2}{2rh} = \dfrac{-r}{2h}$

b) $PV^c = nRT \implies P'V^c + P \cdot cV^{c-1} = 0 \implies P' = -\dfrac{cPV^{c-1}}{V^c} = -\dfrac{cP}{V}$

c) $c^2 = a^2 + b^2 - 2ab\cos\theta$ implies

$$0 = 2aa' + 2b - 2(\cos\theta(a'b + a)) \implies a' = \frac{-2b + 2\cos\theta \cdot a}{2a - 2\cos\theta \cdot b} = \frac{a\cos\theta - b}{a - b\cos\theta}$$

1G. Higher derivatives

1G-1 a) $6 - x^{-3/2}$ b) $\dfrac{-10}{(x+5)^3}$ c) $\dfrac{-10}{(x+5)^3}$ d) 0

1G-2 If $y''' = 0$, then $y'' = c_0$, a constant. Hence $y' = c_0 x + c_1$, where c_1 is some other constant. Next, $y = c_0 x^2/2 + c_1 x + c_2$, where c_2 is yet another constant. Thus, y must be a quadratic polynomial, and any quadratic polynomial will have the property that its third derivative is identically zero.

1G-3

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \implies \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \implies y' = -(b^2/a^2)(x/y)$$

Thus,

$$y'' = -\left(\frac{b^2}{a^2}\right)\left(\frac{y - xy'}{y^2}\right) = -\left(\frac{b^2}{a^2}\right)\left(\frac{y + x(b^2/a^2)(x/y)}{y^2}\right)$$
$$= -\left(\frac{b^4}{y^3a^2}\right)(y^2/b^2 + x^2/a^2) = -\frac{b^4}{a^2y^3}$$

1G-4 $y = (x+1)^{-1}$, so $y^{(1)} = -(x+1)^{-2}$, $y^{(2)} = (-1)(-2)(x+1)^{-3}$, and

$$y^{(3)} = (-1)(-2)(-3)(x+1)^{-4}.$$

The pattern is

$$y^{(n)} = (-1)^n(n!)(x+1)^{-n-1}$$

1G-5 a) $y' = u'v + uv' \implies y'' = u''v + 2u'v' + uv''$

b) Formulas above do coincide with Leibniz's formula for $n = 1$ and $n = 2$. To calculate $y^{(p+q)}$ where $y = x^p(1+x)^q$, use $u = x^p$ and $v = (1+x)^q$. The only term in the Leibniz formula that is not 0 is $\binom{n}{k}u^{(p)}v^{(q)}$, since in all other terms either one factor or the other is 0. If $u = x^p, u^{(p)} = p!$, so

$$y^{(p+q)} = \binom{n}{p}p!q! = \frac{n!}{p!q!} \cdot p!q! = n!$$

1H. Exponentials and Logarithms: Algebra

1H-1 a) To see when $y = y_0/2$, we must solve the equation $\frac{y_0}{2} = y_0e^{kt}$, or $\frac{1}{2} = e^{kt}$.

Take ln of both sides: $-\ln 2 = kt$, from which $t = -\frac{\ln 2}{k}$ ($k < 0$ since stuff is decaying).

b) $y_1 = y_0e^{kt_1}$ by assumption, $\lambda = \frac{-\ln 2}{k}y_0e^{k(t_1+\lambda)} = y_0e^{kt_1} \cdot e^{k\lambda} = y_1 \cdot e^{-\ln 2} = y_1 \cdot \frac{1}{2}$

1H-2 $pH = -\log_{10}[H^+]$; by assumption, $[H^+]_{dil} = \frac{1}{2}[H^+]_{orig}$. Take $-\log_{10}$ of both sides (note that $\log 2 \approx .3$):

$-\log[H^+]_{dil} = \log 2 - \log[H^+]_{orig} \implies pH_{dil} = pH_{orig} + \log_2$.

1H-3 a) $\ln(y+1) + \ln(y-1) = 2x + \ln x$; exponentiating both sides and solving for y:

$(y+1)\cdot(y-1) = e^{2x} \cdot x \implies y^2 - 1 = xe^{2x} \implies y = \sqrt{xe^{2x}+1}$, since $y > 0$.

b) $\log(y+1) - \log(y-1) = -x^2$; exponentiating, $\frac{y+1}{y-1} = 10^{-x^2}$. Solve for y; to simplify the algebra, let $A = 10^{-x^2}$. Crossmultiplying, $y + 1 = Ay - A \implies y = \frac{A+1}{A-1} = \frac{10^{-x^2}+1}{10^{-x^2}-1}$

c) $2\ln y - \ln(y+1) = x$; exponentiating both sides and solving for y:

$\frac{y^2}{y+1} = e^x \implies y^2 - e^xy - e^x = 0 \implies y = \frac{e^x\sqrt{e^{2x}+4e^x}}{2}$, since $y - 1 > 0$.

1H-4 $\frac{\ln a}{\ln b} = c \Rightarrow \ln a = c\ln b \Rightarrow a = e^{c\ln b} = e^{\ln b^c} = b^c$. Similarly, $\frac{\log a}{\log b} = c \Rightarrow a = b^c$.

1H-5 a) Put $u = e^x$ (multiply top and bottom by e^x first): $\frac{u^2+1}{u^2-1} = y$; this gives $u^2 = \frac{y+1}{y-1} = e^{2x}$; taking ln: $2x = \ln(\frac{y+1}{y-1})$, $x = \frac{1}{2}\ln(\frac{y+1}{y-1})$

b) $e^x + e^{-x} = y$; putting $u = e^x$ gives $u + \frac{1}{u} = y$; solving for u gives $u^2 - yu + 1 = 0$ so that $u = \frac{y \pm \sqrt{y^2-4}}{2} = e^x$; taking ln: $x = \ln(\frac{y \pm \sqrt{y^2-4}}{2})$

1H-6 $A = \log e \cdot \ln 10 = \ln(10^{\log e}) = \ln(e) = 1$; similarly, $\log_b a \cdot \log_a b = 1$

1H-7 a) If I_1 is the intensity of the jet and I_2 is the intensity of the conversation, then

$$\log_{10}(I_1/I_2) = \log_{10}\left(\frac{I_1/I_0}{I_2/I_0}\right) = \log_{10}(I_1/I_0) - \log_{10}(I_2/I_0) = 13 - 6 = 7$$

Therefore, $I_1/I_2 = 10^7$.

b) $I = C/r^2$ and $I = I_1$ when $r = 50$ implies

$$I_1 = C/50^2 \implies C = I_1 50^2 \implies I = I_1 50^2/r^2$$

This shows that when $r = 100$, we have $I = I_1 50^2/100^2 = I_1/4$. It follows that

$$10\log_{10}(I/I_0) = 10\log_{10}(I_1/4I_0) = 10\log_{10}(I_1/I_0) - 10\log_{10}4 \approx 130 - 6.0 \approx 124$$

The sound at 100 meters is 124 decibels.

The sound at 1 km has 1/100 the intensity of the sound at 100 meters, because $100m/1km = 1/10$.

$$10\log_{10}(1/100) = 10(-2) = -20$$

so the decibel level is $124 - 20 = 104$.

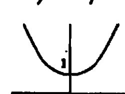
1I. Exponentials and Logarithms: Calculus

1I-1 a) $(x+1)e^x$ b) $4xe^{2x}$ c) $(-2x)e^{-x^2}$ d) $\ln x$ e) $2/x$ f) $2(\ln x)/x$ g) $4xe^{2x^2}$

h) $(x^x)' = (e^{x\ln x})' = (x\ln x)'e^{x\ln x} = (\ln x + 1)e^{x\ln x} = (1+\ln x)x^x$

i) $(e^x - e^{-x})/2$ j) $(e^x + e^{-x})/2$ k) $-1/x$ l) $-1/x(\ln x)^2$ m) $-2e^x/(1+e^x)^2$

1I-2



(even)

1I-3 a) As $n \to \infty$, $h = 1/n \to 0$.

$$n\ln(1+\frac{1}{n}) = \frac{\ln(1+h)}{h} = \frac{\ln(1+h) - \ln(1)}{h} \xrightarrow[h\to 0]{} \frac{d}{dx}\ln(1+x)\Big|_{x=0} = 1$$

Therefore,

$$\lim_{n\to\infty} n\ln(1+\frac{1}{n}) = 1$$

b) Take the logarithm of both sides. We need to show

$$\lim_{n\to\infty} \ln(1+\frac{1}{n})^n = \ln e = 1$$

But

$$\ln(1+\frac{1}{n})^n = n\ln(1+\frac{1}{n})$$

so the limit is the same as the one in part (a).

1I-4 a)

$$\left(1+\frac{1}{n}\right)^{3n} = \left(\left(1+\frac{1}{n}\right)^{n}\right)^{3} \longrightarrow e^3 \text{ as } n \to \infty,$$

b) Put $m = n/2$. Then

$$\left(1+\frac{2}{n}\right)^{5n} = \left(1+\frac{1}{m}\right)^{10m} = \left(\left(1+\frac{1}{m}\right)^{m}\right)^{10} \longrightarrow e^{10} \text{ as } m \to \infty$$

c) Put $m = 2n$. Then

$$\left(1+\frac{1}{2n}\right)^{5n} = \left(1+\frac{1}{m}\right)^{5m/2} = \left(\left(1+\frac{1}{m}\right)^{m}\right)^{5/2} \longrightarrow e^{5/2} \text{ as } m \to \infty$$

1J. Trigonometric functions

1J-1 a) $10x\cos(5x^2)$ b) $6\sin(3x)\cos(3x)$ c) $-2\sin(2x)/\cos(2x) = -2\tan(2x)$

d) $-2\sin x/(2\cos x) = -\tan x$. (Why did the factor 2 disappear? Because $\ln(2\cos x) = \ln 2 + \ln(\cos x)$, and the derivative of the constant $\ln 2$ is zero.)

e) $\dfrac{x\cos x - \sin x}{x^2}$ f) $-(1+y')\sin(x+y)$ g) $-\sin(x+y)$ h) $2\sin x\cos x e^{\sin^2 x}$

i) $\dfrac{(x^2\sin x)'}{x^2\sin x} = \dfrac{2x\sin x + x^2\cos x}{x^2\sin x} = \dfrac{2}{x} + \cot x$. Alternatively,

$$\ln(x^2\sin x) = \ln(x^2) + \ln(\sin x) = 2\ln x + \ln\sin x$$

Differentiating gives $\quad \dfrac{2}{x} + \dfrac{\cos x}{\sin x} = \dfrac{2}{x} + \cot x$

j) $2e^{2x}\sin(10x) + 10e^{2x}\cos(10x)$ k) $6\tan(3x)\sec^2(3x) = 6\sin x/\cos^3 x$

l) $-x(1-x^2)^{-1/2}\sec(\sqrt{1-x^2})\tan(\sqrt{1-x^2})$

m) Using the chain rule repeatedly and the trigonometric double angle formulas,

$$\begin{aligned}(\cos^2 x - \sin^2 x)' &= -2\cos x\sin x - 2\sin x\cos x = -4\cos x\sin x;\\ (2\cos^2 x)' &= -4\cos x\sin x;\\ (\cos(2x))' &= -2\sin(2x) = -2(2\sin x\cos x).\end{aligned}$$

The three functions have the same derivative, so they differ by constants. And indeed,

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1, \qquad (\text{using } \sin^2 x = 1 - \cos^2 x).$$

n)

$$5(\sec(5x)\tan(5x))\tan(5x) + 5(\sec(5x)(\sec^2(5x)) = 5\sec(5x)(\sec^2(5x) + \tan^2(5x))$$

Other forms: $\quad 5\sec(5x)(2\sec^2(5x) - 1); \qquad 10\sec^3(5x) - 5\sec(5x)$

o) 0 because $\sec^2(3x) - \tan^2(3x) = 1$, a constant — or carry it out for practice.

p) Successive use of the chain rule:

$$
\begin{aligned}
(\sin(\sqrt{x^2+1}))' &= \cos(\sqrt{x^2+1}) \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x \\
&= \frac{x}{\sqrt{x^2+1}} \cos(\sqrt{x^2+1})
\end{aligned}
$$

q) Chain rule several times in succession:

$$
\begin{aligned}
(\cos^2\sqrt{1-x^2})' &= 2\cos\sqrt{1-x^2} \cdot (-\sin\sqrt{1-x^2}) \cdot \frac{-x}{\sqrt{1-x^2}} \\
&= \frac{x}{\sqrt{1-x^2}} \sin(2\sqrt{1-x^2})
\end{aligned}
$$

r) Chain rule again:

$$
\begin{aligned}
\left(\tan^2(\frac{x}{x+1})\right) &= 2\tan(\frac{x}{x+1}) \cdot \sec^2(\frac{x}{x+1}) \cdot \frac{x+1-x}{(x+1)^2} \\
&= \frac{2}{(x+1)^2} \tan(\frac{x}{x+1}) \sec^2(\frac{x}{x+1})
\end{aligned}
$$

1J-2 Because $\cos(\pi/2) = 0$,

$$
\lim_{x\to\pi/2} \frac{\cos x}{x-\pi/2} = \lim_{x\to\pi/2} \frac{\cos x - \cos(\pi/2)}{x - \pi/2} = \frac{d}{dx}\cos x\Big|_{x=\pi/2} = -\sin x\Big|_{x=\pi/2} = -1
$$

1J-3 a) $(\sin(kx))' = k\cos(kx)$. Hence

$$
(\sin(kx))'' = (k\cos(kx))' = -k^2\sin(kx).
$$

Similarly, differentiating cosine twice switches from sine and then back to cosine with only one sign change, so

$$
(\cos(kx)'' = -k^2\cos(kx)
$$

Therefore,

$$
\sin(kx)'' + k^2\sin(kx) = 0 \text{ and } \cos(kx)'' + k^2\cos(kx) = 0
$$

Since we are assuming $k > 0$, $k = \sqrt{a}$.

b) This follows from the linearity of the operation of differentiation. With $k^2 = a$,

$$
\begin{aligned}
&(c_1\sin(kx) + c_2\cos(kx))'' + k^2(c_1\sin(kx) + c_2\cos(kx)) \\
&\quad = c_1(\sin(kx))'' + c_2(\cos(kx))'' + k^2c_1\sin(kx) + k^2c_2\cos(kx) \\
&\quad = c_1[(\sin(kx))'' + k^2\sin(kx)] + c_2[(\cos(kx))'' + k^2\cos(kx)] \\
&\quad = c_1\cdot 0 + c_2\cdot 0 = 0
\end{aligned}
$$

c) Since ϕ is a constant, $d(kx+\phi)/dx = k$, and $(\sin(kx+\phi)' = k\cos(kx+\phi)$,

$$(\sin(kx+\phi)'' = (k\cos(kx+\phi))' = -k^2\sin(kx+\phi)$$

Therefore, if $a = k^2$,

$$(\sin(kx+\phi)'' + a\sin(kx+\phi) = 0$$

d) The sum formula for the sine function says

$$\sin(kx+\phi) = \sin(kx)\cos(\phi) + \cos(kx)\sin(\phi)$$

In other words

$$\sin(kx+\phi) = c_1\sin(kx) + c_2\cos(kx)$$

with $c_1 = \cos(\phi)$ and $c_2 = \sin(\phi)$.

1J-4 a) The Pythagorean theorem implies that

$$c^2 = \sin^2\theta + (1-\cos\theta)^2 = \sin^2\theta + 1 - 2\cos\theta + \cos^2\theta = 2 - 2\cos\theta$$

Thus,

$$c = \sqrt{2-2\cos\theta} = 2\sqrt{\frac{1-\cos\theta}{2}} = 2\sin(\theta/2)$$

b) Each angle is $\theta = 2\pi/n$, so the perimeter of the n-gon is

$$n\sin(2\pi/n)$$

As $n \to \infty$, $h = 2\pi/n$ tends to 0, so

$$n\sin(2\pi/n) = \frac{2\pi}{h}\sin h = 2\pi\frac{\sin h - \sin 0}{h} \to 2\pi\frac{d}{dx}\sin x|_{x=0} = 2\pi\cos x|_{x=0} = 2\pi$$

2. Applications of Differentiation

2A. Approximation

2A-1 $\frac{d}{dx}\sqrt{a+bx} = \frac{b}{2\sqrt{a+bx}} \Rightarrow f(x) \approx \sqrt{a} + \frac{b}{2\sqrt{a}}x$ by formula.

By algebra: $\sqrt{a+bx} = \sqrt{a}\sqrt{1+\frac{bx}{a}} \approx \sqrt{a}(1+\frac{bx}{2a})$, same as above.

2A-2 $D(\frac{1}{a+bx}) = \frac{-b}{(a+bx)^2} \Rightarrow f(x) \approx \frac{1}{a} - \frac{b}{a^2}x$; OR: $\frac{1}{a+bx} = \frac{1/a}{1+b/ax} \approx \frac{1}{a}(1-\frac{b}{a}x)$.

2A-3 $D(\frac{(1+x)^{3/2}}{1+2x}) = \frac{(1+2x)\cdot\frac{3}{2}\cdot(1+x)^{3/2} - (1+x)^{3/2}\cdot 2}{(1+2x)^2} \Rightarrow f'(0) = -\frac{1}{2}$

$\Rightarrow f(x) \approx 1 - \frac{1}{2}x$; OR, by algebra, $\frac{(1+x)^{3/2}}{1+2x} \approx (1+\frac{3}{2}x)(1-2x) \approx 1 - \frac{1}{2}x$.

2A-4 Put $\frac{h}{R} = \epsilon$; then $w = \frac{g}{(1+\epsilon)^2} \approx g(1-\epsilon)^2 \approx g(1-2\epsilon) = g(1-\frac{2h}{R})$.

2A-5 A reasonable assumption is that w is propotional to volume v, which is in turn proportional to the *cube* of a linear dimension, i.e., a given person remains similar to him/herself, for *small* weight changes.) Thus $w = Ch^3$; since 5 feet = 60 inches, we get

$$\frac{w(60+\epsilon)}{w(60)} = \frac{C(60+\epsilon)^3}{C(60)^3} = (1+\frac{\epsilon}{60})^3 \Rightarrow w(60+\epsilon) \approx w(60)\cdot(1+\frac{3\epsilon}{60}) \approx 120\cdot(1+\frac{1}{20}) \approx 126.$$

[Or you can calculate the linearization of $w(h)$ arround $h = 60$ using derivatives, and using the value $w(60)$ to determine C. getting $w(h) \approx 120 + 6(h-60)$

2A-6 $\tan\theta = \frac{\sin\theta}{\cos\theta} \approx \frac{\theta}{1-\theta^2/2} \approx \theta(1+\theta^2//2) \approx \theta$

2A-7 $\frac{\sec x}{\sqrt{1-x^2}} = \frac{1}{\cos x\sqrt{1-x^2}} \approx \frac{1}{(1-\frac{1}{2}x^2)(1-\frac{1}{2}x^2)} \approx \frac{1}{1-x^2} \approx 1+x^2$

2A-8 $\frac{1}{1-x} = \frac{1}{1-(\frac{1}{2}+\Delta x)} = \frac{1}{\frac{1}{2}-\Delta x} = \frac{2}{1-2\Delta x}$

$$\approx 2(1+2\Delta x+4(\Delta x)^2) \approx 2+4(x-\tfrac{1}{2})+8(x-\tfrac{1}{2})^2$$

2A-10 $y = (1+x)^r, y' = r(1+x)^{r-1}, y'' = r(r-1)(1+x)^{r-2}$

Therefore $y(0) = 1, y'(0) = r, y''(0) = r(r-1)$, giving $(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2$.

2A-11 $pv^k = c \Rightarrow p = cv^{-k} = c((v_0+\Delta v)^{-k} = cv_0{}^{-k}(1+\frac{\Delta v}{v_0})^{-k}$

$$\approx \frac{c}{v_0{}^k}(1 - k\frac{\Delta v}{v_0} + \frac{k(k+1)}{2}(\frac{\Delta v}{v_0})^2)$$

2A-12 a) $\frac{e^x}{1-x} \approx (1+x+\frac{x^2}{2})(1+x+x^2) \approx 1+2x+\frac{5}{2}x^2$

b) $\dfrac{\ln(1+x)}{xe^x} \approx \dfrac{x}{x(1+x)} \approx 1-x$

c) $e^{-x^2} \approx 1-x^2$ [Substitute into $e^x \approx 1+x$]

d) $\ln(\cos x) \approx \ln(1-\frac{x^2}{2}) \approx -\frac{x^2}{2}$ [since $\ln(1+h) \approx h$]

e) $x\ln x = (1+h)\ln(1+h) \approx (1+h)(h-\frac{h^2}{2}) \approx h+\frac{h^2}{2} \Rightarrow x\ln x \approx (x-1)+\frac{(x-1)^2}{2}$

2A-13 Finding the linear and quadratic approximation

a) $2x$ (both linear and quadratic)

b) $1,\ 1-2x^2$

c) $1,\ 1+x^2/2$ (Use $(1+u)^{-1} \approx 1-u$ with $u = x^2/2$:

$$\sec x = 1/\cos x \approx 1/(1-x^2/2) = (1-x^2/2)^{-1} \approx 1+x^2/2$$

d) $1,\ 1+x^2$

e) Use $(1+u)^{-1} \approx 1-u+u^2$:

$$(a+bx)^{-1} = a^{-1}(1+(bx/a))^{-1} \approx a^{-1}(1-bx/a+(bx/a)^2)$$

Linear approximation: $(1/a)-(b/a^2)x$

Quadratic approximation: $(1/a)-(b/a^2)x+(b^2/a^3)x^2$

f) $f(x) = 1/(a+bx)$ so that $f'(1) = -b(a+b)^{-2}$ and $f''(1) = 2b^2/(a+b)^{-3}$. We need to assume that these numbers are defined, in other words that $a+b \neq 0$. Then the linear approximation is

$$1/(a+b)-(b/(a+b)^2)(x-1)$$

and the quadratic approximation is

$$1/(a+b)-(b/(a+b)^2)(x-1)+(b/(a+b)^3)(x-1)^2$$

Method 2: Write

$$1/(a+bx) = 1/(a+b+b(x-1))$$

Then use the expansion of problem (e) with $a+b$ in place of a and b in place of b and $(x-1)$ in place of x. The requirement $a \neq 0$ in (e) corresponds to the restriction $a+b \neq 0$ in (f).

2A-15 $f(x) = \cos(3x)$, $\quad f'(x) = -3\sin(3x)$, $\quad f''(x) = -9\cos(3x)$. Thus,

$$f(0) = 1, \quad f(\pi/6) = \cos(\pi/2) = 0, \quad f(\pi/3) = \cos\pi = -1$$
$$f'(0) = -3\sin 0 = 0, \quad f'(\pi/6) = -3\sin(\pi/2) = -3, \quad f'(\pi/3) = -3\sin\pi = 0$$
$$f''(0) = -9, \quad f''(\pi/6) = 0, \quad f''(\pi/3) = 9$$

Using these values, the linear and quadratic approximations are respectively:

$$\begin{aligned} \text{for } x \approx 0: &\quad f(x) \approx 1 \text{ and } f(x) \approx 1-(9/2)x^2 \\ \text{for } x \approx \pi/6: &\quad \text{both are } f(x) \approx -3(x-\pi/6) \\ \text{for } x \approx \pi/3: &\quad f(x) \approx -1 \text{ and } f(x) \approx -1+(9/2)(x-\pi/3)^2 \end{aligned}$$

2A-16 a) The law of cosines says that for a triangle with sides a, b, and c, with θ opposite the side of length c,

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Apply it to one of the n triangles with vertex at the origin: $a = b = 1$ and $\theta = 2\pi/n$. So the formula is

$$c = \sqrt{2 - 2\cos(2\pi/n)}$$

b) The perimeter is $n\sqrt{2 - 2\cos(2\pi/n)}$. The quadratic approximation to $\cos\theta$ near 0 is

$$\cos\theta \approx 1 - \theta^2/2$$

Therefore, as $n \to \infty$ and $\theta = 2\pi/n \to 0$,

$$n\sqrt{2 - 2\cos(2\pi/n)} \approx n\sqrt{2 - 2(1 - (1/2)(2\pi/n)^2)} = n\sqrt{(2\pi/n)^2} = n(2\pi/n) = 2\pi$$

In other words,

$$\lim_{n\to\infty} n\sqrt{2 - 2\cos(2\pi/n)} = 2\pi,$$

the circumference of the circle of radius 1.

2B. Curve Sketching

2B-1 a) $y = x^3 - 3x + 1$, $y' = 3x^2 - 3 = 3(x-1)(x+1)$. $y' = 0 \implies x = \pm 1$.

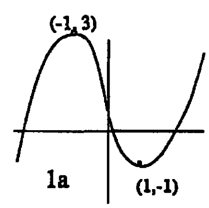
Endpoint values: $y \to -\infty$ as $x \to -\infty$, and $y \to \infty$ as $x \to \infty$.

Critical values: $y(-1) = 3$, $y(1) = -1$.

Increasing on: $-\infty < x < -1$, $1 < x < \infty$.

Decreasing on: $-1 < x < 1$.

Graph: $(-\infty, -\infty) \nearrow (-1, 3) \searrow (1, -1) \nearrow (\infty, \infty)$, crossing the x-axis three times.



b) $y = x^4 - 4x + 1$, $y' = x^3 - 4$. $y' = 0 \implies x = 4^{1/3}$.

Increasing on: $4^{1/3} < x < \infty$; decreasing on: $-\infty < x < 4^{1/3}$.

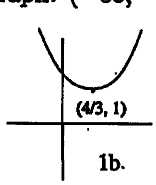
Endpoint values: $y \to \infty$ as $x \to \pm\infty$; critical value: $y(4^{1/3}) = 1$.

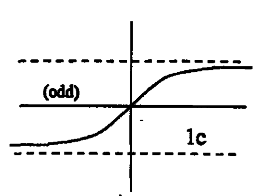
Graph: $(-\infty, \infty) \searrow (4^{1//3}, 1) \nearrow (\infty, \infty)$, never crossing the x-axis. (See below.)

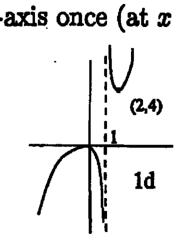
c) $y'(x) = 1/(1 + x^2)$ and $y(0) = 0$. By inspection, $y' > 0$ for all x, hence always increasing.

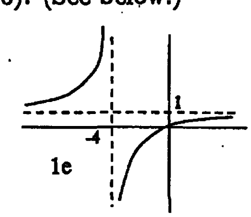
Endpoint values: $y \to c$ as $x \to \infty$ and by symmetry $y \to -c$ as $x \to -\infty$. (But it is not clear at this point in the course whether $c = \infty$ or some finite value. It turns out (in Lecture 26) that $y \to c = \pi/2$.

Graph: $(-\infty, -c) \nearrow (\infty, c)$, crossing the x-axis once (at $x = 0$). (See below.)









d) $y = x^2/(x-1)$, $y' = (2x(x-1)-x^2)/(x-1)^2 = (x^2-2x)/(x-1)^2 = (x-2)x/(x-1)^2$.

Endpoint values: $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$.

Singular values: $y(1^+) = +\infty$ and $y(1^-) = -\infty$.

Critical values: $y(0) = 0$ and $y(2) = 4$.

New feature: Pay attention to sign changes in the denominator of y'.

Increasing on: $-\infty < x < 0$ and $2 < x < \infty$

Decreasing on: $0 < x < 1$ and $1 < x < 2$

Graph: $(-\infty,-\infty) \nearrow (0,0) \searrow (1,-\infty) \uparrow (1,\infty) \searrow (2,4) \nearrow (\infty,\infty)$, crossing the x-axis once (at $x = 0$).

Commentary on singularities: Look out for sign changes both where y' is zero and also where y' is undefined: $y' = 0$ indicates a possible sign change in the numerator and y' undefined indicates a possible sign change in the denominator. In this case there was no sign change in y' at $x = 1$, but there would have been a sign change, if there had been an odd power of $(x-1)$ in the denominator.

e) $y = x/(x+4)$, $y' = ((x+4)-x)/(x+4)^2 = 4/(x+4)^2$. No critical points.

Endpoint values: $y \to 1$ as $x \to \pm\infty$.

Increasing on: $-4 < x < \infty$.

Decreasing on: $-\infty < x < -4$.

Singular values: $y(-4^+) = -\infty$, $y(-4^-) = +\infty$.

Graph: $(-\infty, 1) \nearrow (-4,\infty) \downarrow (-4,-\infty) \nearrow (\infty, 1)$, crossing the x-axis once (at $x = 0$).

f) $y = \sqrt{x+1}/(x-3)$, $y' = -(1/2)(x+5)(x+1)^{-1/2}(x-3)^{-2}$ No critical points because $x = -5$ is outside of the domain of definition, $x \geq -1$.

Endpoint values: $y(-1) = 0$, and as $x \to \infty$,

$$y = \frac{1 + \frac{1}{x}}{\sqrt{x} - \frac{3}{\sqrt{x}}} \to \frac{1}{\infty} = 0$$

Singular values: $y(3^+) = +\infty$, $y(3^-) = -\infty$.

Increasing on: nowhere

Decreasing on: $-1 < x < 3$ and $3 < x < \infty$.

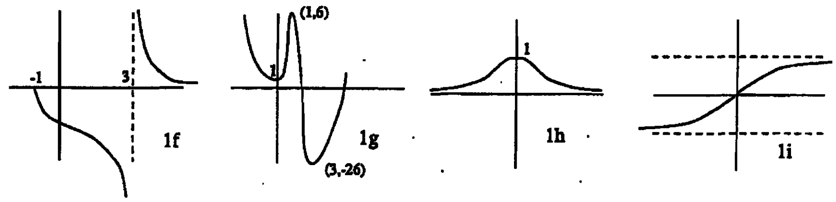
Graph: $(-1,0) \searrow (3,-\infty) \uparrow (3,\infty) \searrow (\infty, 0)$, crossing the x-axis once (at $x = -1$).

g) $y = 3x^4 - 16x^3 + 18x^2 + 1$, $y' = 12x^3 - 48x^2 + 36x = 12x(x-1)(x-3)$. $y' = 0 \implies x = 0, 1, 3$.

Endpoint values: $y \to \infty$ as $x \to \pm\infty$.

Critical values: $y(0) = 1$, $y(1) = 6$, and $y(3) = -188$.

Increasing on: $0 < x < 1$ and $3 < x < \infty$.



Decreasing on: $-\infty < x < 0$ and $1 < x < 3$.

Graph: $(-\infty, \infty) \searrow (0,1) \nearrow (1,6) \searrow (3,-188) \nearrow (\infty, \infty)$, crossing the x-axis once.

h) $y = e^{-x^2}$, $y' = -2xe^{-x^2}$. $y' = 0 \implies x = 0$.

Endpoint values: $y \to 0$ as $x \to \pm\infty$.

Critical value: $y(0) = 1$.

Increasing on: $-\infty < x < 0$

Decreasing on: $0 < x < \infty$

Graph: $(-\infty, 0) \nearrow (0,1) \searrow (\infty, 0)$, never crossing the x-axis. (The function is even.)

i) $y' = e^{-x^2}$ and $y(0) = 0$. Because y' is even and $y(0) = 0$, y is odd. No critical points.

Endpoint values: $y \to c$ as $x \to \infty$ and by symmetry $y \to -c$ as $x \to -\infty$. It is not clear at this point in the course whether c is finite or infinite. But we will be able to show that c is finite when we discuss improper integrals in Unit 6. (Using a trick with iterated integrals, a subject in 18.02, one can show that $c = \sqrt{\pi}/2$.)

Graph: $(-\infty, -c) \nearrow (\infty, c)$, crossing the x-axis once (at $x = 0$).

2B-2 a) One inflection point at $x = 0$. ($y'' = 6x$)

b) No inflection points. $y'' = 3x^2$, so the function is convex. $x = 0$ is not a point of inflection because $y'' > 0$ on both sides of $x = 0$.

c) Inflection point at $x = 0$. ($y'' = -2x/(1+x^2)^2$)

d) No inflection points. Reasoning: $y'' = 2/(x-1)^3$. Thus $y'' > 0$ and the function is concave up when $x > 1$, and $y'' < 0$ and the function is concave down when $x < 1$. But $x = 1$ is not called an inflection point because the function is not continuous there. In fact, $x = 1$ is a singular point.

e) No inflection points. $y'' = -8/(x+1)^3$. As in part (d) there is a sign change in y'', but at a singular point not an inflection point.

f) $y'' = -(1/2)[(x+1)(x-3) - (1/2)(x+5)(x-3) - 2(x+5)(x+1)](x+1)^{-3/2}(x-3)^3$
$= -(1/2)[-(3/2)x^2 - 15x - 11/2](x+1)^{-3/2}(x-3)^3$

Therefore there are two inflection points, $x = (-30 \pm \sqrt{768})/6$, $\approx 9.6, .38$.

g) $y'' = 12(3x^2 - 8x + 36)$. Therefore there are no inflection points. The quadratic equation has no real roots.

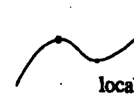
h) $y'' = (-2 + 4x^2)e^{-x^2}$. Therefore there are two inflection points at $x = \pm 1/\sqrt{2}$.

i) One inflection point at $x = 0$. ($y'' = -2xe^{-x^2}$)

2B-3 a) $y' = 3x^2 + 2ax + b$. The roots of the quadratic polynomial are distinct real numbers if the discriminant is positive. (The ***discriminant*** is defined as the number under the square root in the quadratic formula.) Therefore there are distinct real roots if and only if

$$(2a)^2 - 4(3)b > 0, \qquad \text{or} \qquad a^2 - 3b > 0.$$

From the picture, since $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$, the larger root of $3x^2 + 2ax + b = 0$ (with the plus sign in the quadratic formula) must be the local min, and the smaller root must be the local max.



Since y <<-1 when x << -1 and y >> 1 when x >> 1, the local max. is to the left of the local min.

b) $y'' = 6x + 2a$, so the inflection point is at $-a/3$. Therefore the condition $y' < 0$ at the inflection point is

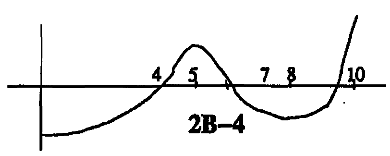
$$y'(-a/3) = 3(-a/3)^2 + 2a(-a/3) + b = -a^2/3 + b < 0,$$

which is the same as

$$a^2 - 3b > 0.$$

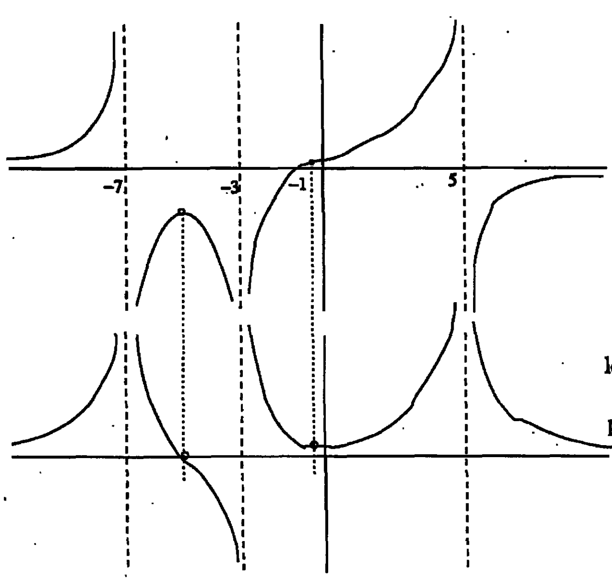
If $y' < 0$ at some point x_0, then the function is decreasing at that point. But $y \to \infty$ as $x \to \infty$, so there must be a local minimum at a point $x > x_0$. Similarly, since $y \to -\infty$ as $x \to -\infty$, there must be a local maximum at a point $x < x_0$.

Comment: We evaluate y' at the inflection point of y ($x = -a/3$) since we are trying to decide (cf. part (b)) whether y' is ever negative. To do this, we find the minimum of y' (which occurs where $y'' = 0$).



2B–4

Max is at x = 5 or x = 10;
Min is at x = 0 or x = 8.



2B–5

Graph of function

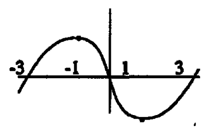
Graph of derivative; note that local maximum point above corresponds to zero below; point of inflection above corresponds to local minimum below.

2B-6 a) Try $y' = (x+1)(x-1) = x^2 - 1$. Then $y = x^3/3 - x + c$. The constant c won't matter so set $c = 0$. It's also more convenient to multiply by 3:

$$y = x^3 - 3x$$

b) This is an odd function with local min and max: $y(1) = -2$ and $y(-1) = 2$. The endpoints values are $y(3) = 18$ and $y(-3) = -18$. It is very steep: $y'(3) = 8$.

c)



2B-7 a) $f'(a) = \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$

If y increasing then $\left.\begin{matrix} \Delta y > 0 \Rightarrow \Delta x > 0 \\ \Delta y < 0 \Rightarrow \Delta x < 0 \end{matrix}\right\}$. So in both cases $\dfrac{\Delta y}{\Delta x} > 0$.

Therefore, $\lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} \geq 0$.

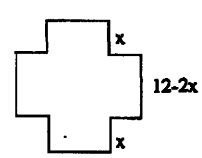
b) Proof breaks down at the last step. Namely, $\dfrac{\Delta y}{\Delta x} > 0$ doesn't imply $\lim_{\Delta x \to 0} \dfrac{\Delta x}{\Delta y} > 0$

[Limits don't preserve strict inequalities, only weak ones. For example, $u^2 > 0$ for $u \neq 0$, but $\lim_{u \to 0} u^2 = 0 \geq 0$, not > 0.]

Counterexample: $f(x) = x^3$ is increasing for all x, but $f'(0) = 0$.

c) Use $f(a) \geq f(x)$ to show that $\lim_{\Delta x \to 0+} \Delta y/\Delta x \leq 0$ and $\lim_{\Delta x \to 0-} \Delta y/\Delta x \geq 0$. Since the left and right limits are equal, the derivative must be zero.

2C. Max-min problems

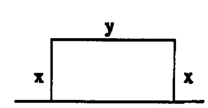


2C-1 The base of the box has sidelength $12 - 2x$ and the height is x, so the volume is $V = x(12-2x)^2$.

At the endpoints $x = 0$ and $x = 6$, the volume is 0, so the maximum must occur in between at a critical point.

$$V' = (12-2x)^2 + x(2)(12-2x)(-2) = (12-2x)(12-2x-4x) = (12-2x)(12-6x).$$

It follows that $V' = 0$ when $x = 6$ or $x = 2$. At the endpoints $x = 0$ and $x = 6$ the volume is 0, so the maximum occurs when $x = 2$.



2C-2 We want to minimize the fence length $L = 2x + y$, where the variables x and y are related by $xy = A = 20,000$.

Choosing x as the independent variable, we have $y = A/x$, so that $L = 2x + A/x$. At the endpoints $x = 0$ and $x = \infty$ (it's a long barn), we get $L = \infty$, so the minimumof L must occur at a critical point.

$$L' = 2 - \frac{A}{x^2}; \quad L = 0 \implies x^2 = \frac{A}{2} = 10,000 \implies x = 100 \text{ feet}$$

2C-3 We have $y = (a - x)/2$, so $xy = x(a - x)/2$. At the endpoints $x = 0$ and $x = a$, the product xy is zero (and beyond it is negative). Therefore, the maximum occurs at a critical point. Taking the derivative,

$$\frac{d}{dx}\frac{x(a-x)}{2} = \frac{a-2x}{2}; \quad \text{this is 0 when } x = a/2.$$

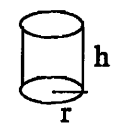
2C-4 If the length is y and the cross-section is a square with sidelength x, then $4x+y = 108$. Therxefore the volume is $V = x^2y = 108x^2 - 4x^3$. Find the critical points:

$$(108x^2 - 4x^3)' = 216x - 12x^2 = 0 \implies x = 18 \text{ or } x = 0.$$

The critical point $x = 18$ (3/2 ft.) corresponds to the length $y = 36$ (3 ft.), giving therefore a volume of $(3/2)^2(3) = 27/4 = 6.75$ cubic feet.

The endpoints are $x = 0$, which gives zero volume, and when $x = y$, i.e., $x = 9/5$ feet, which gives a volume of $(9/5)^3$ cubic feet, which is less than 6 cubic feet. So the critical point gives the maximum volume.

2C-5 We let r = radius of bottom and h = height, then the volume is $V = \pi r^2 h$, and the area is $A = \pi r^2 + 2\pi rh$.



Using r as the independent variable, we have using the above formulas,

$$h = \frac{A - \pi r^2}{2\pi r}, \qquad V = \pi r^2 h = \left(\frac{A}{2}r - \frac{\pi}{2}r^3\right); \qquad \frac{dV}{dr} = \frac{A}{2} - \frac{3\pi}{2}r^2.$$

Therefore, $dV/dr = 0$ implies $A = 3\pi r^2$, from which $h = \dfrac{A - \pi r^2}{2\pi r} = r$.

Checking the endpoints, at one $h = 0$ and $V = 0$; at the other, $\lim\limits_{r\to 0+} V = 0$ (using the expression above for V in terms of r); thus the critical point must occur at a maximum.

(Another way to do this problem is to use implicit differentiation with respect to r. Briefly, since A is fixed, $dA/dr = 0$, and therefore

$$\frac{dA}{dr} = 2\pi r + 2\pi h + 2\pi r h' = 0 \implies h' = -\frac{r+h}{r};$$
$$\frac{dV}{dr} = 2\pi rh + \pi r^2 h' = 2\pi rh - \pi r(r+h) = \pi r(h - r).$$

It follows that $V' = 0$ when $r = h$ or $r = 0$, and the latter is a rejected endpoint.

2C-6 To get max and min of $y = x(x+1)(x-1) = x^3 - x$, first find the critical points:

$$y' = 3x^2 - 1 = 0 \quad \text{if} \quad x = \pm\frac{1}{\sqrt{3}};$$

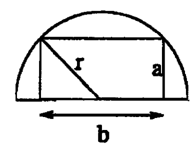
$$y(\frac{1}{\sqrt{3}}) = \frac{1}{\sqrt{3}}(-\frac{2}{3}) = \frac{-2}{3\sqrt{3}}, \text{ rel. min.} \qquad y(-\frac{1}{\sqrt{3}}) = -\frac{1}{\sqrt{3}}(-\frac{2}{3}) = \frac{2}{3\sqrt{3}}, \text{ rel. max.}$$

Check endpoints: $y(2) = 6 \Rightarrow 2$ is absolute max.; $y(-2) = -6 \Rightarrow -2$ is absolute min.

(This is an *endpoint problem.* The endpoints should be tested unless the physical or geometric picture already makes clear whether the max or min occurs at an endpoint.)

2C-7 Let r be the radius, which is fixed. Then the height a of the rectangle is in the interval $0 \le a \le r$. Since $b = 2\sqrt{r^2 - a^2}$, the area A is given in terms of a by

$$A = 2a\sqrt{r^2 - a^2}.$$



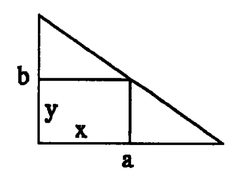
The value of A at the endpoints $a = 0$ and $a = r$ is zero, so the maximum occurs at a critical point in between.

$$\frac{dA}{da} = 2\sqrt{r^2 - a^2} - \frac{2a^2}{\sqrt{r^2 - a^2}} = \frac{2(r^2 - a^2) - 2a^2}{\sqrt{r^2 - a^2}}$$

Thus $dA/da = 0$ implies $2r^2 = a^2$, from which we get $a = \frac{r}{\sqrt{2}}$, $b = r\sqrt{2}$.
(We use the positive square root since $a \ge 0$. Note that $b = 2a$ and $A = r^2$.)

2C-8 a) Letting a and b be the two legs and x and y the sides of the rectangle, we have $y = -(b/a)(x - a)$ and the area $A = xy = (b/a)x(a - x)$. The area is zero at the two ends $x = 0$ and $x = a$, so the maximum occurs in between at a critical point:

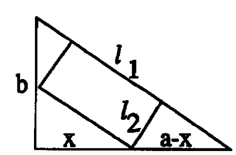
$$A' = (b/a)((a - x) - x); \quad = 0 \text{ if } x = a/2.$$



Thus $y = (b/a)(a - x) = b/2$ and $A = ab/4$.

b) This time let x be the point shown on the accompanying figure; using similar triangles, the sides of the rectangle are

$$\ell_1 = \frac{x}{a}\sqrt{a^2 + b^2} \text{ and } \ell_2 = \frac{b}{\sqrt{a^2 + b^2}}(a - x)$$



Therefore the area is

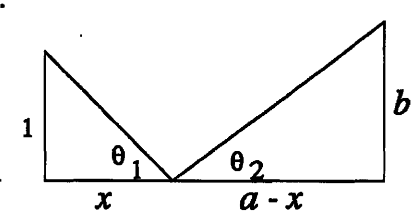
$$A = \ell_1\ell_2 = (b/a)x(a - x)$$

This is the same formula for area as in part (a), so the largest area is the same, occurring when $x = a/2$, and the two maximal rectangles both have the same area; they have different dimensions though, since in the present case, one side length is half the hypotenuse:

$$\ell_1 = \sqrt{a^2 + b^2}/2 \text{ and } \ell_2 = ab/2\sqrt{a^2 + b^2}.$$

2C-9 The distance is

$$L = \sqrt{x^2 + 1} + \sqrt{(a - x)^2 + b^2}$$



The endpoint values are $x \to \pm\infty$, for $L \to \infty$, so the minimum value is at a critical point.

$$L' = \frac{x}{\sqrt{x^2 + 1}} - \frac{a - x}{\sqrt{(a - x)^2 + b^2}} = \frac{x\sqrt{(a - x)^2 + b^2} - (a - x)\sqrt{x^2 + 1}}{\sqrt{x^2 + 1}\sqrt{(a - x)^2 + b^2}}$$

Thus $L' = 0$ implies (after squaring both sides),

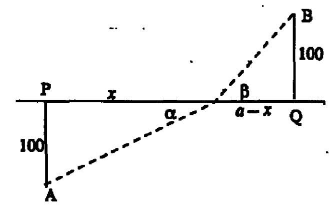
$$x^2((a - x)^2 + b^2) = (a - x)^2(x^2 + 1), \quad \text{or} \quad x^2b^2 = (a - x)^2 \quad \text{or} \quad bx = (a - x);$$

we used the positive square roots since both sides must be positive. Rewriting the above,

$$\frac{b}{a-x} = \frac{1}{x}, \qquad \text{or} \qquad \tan\theta_1 = \tan\theta_2.$$

Thus $\theta_1 = \theta_2$: the angle of incidence equals the angle of reflection.

2C-10 The total time is



$$T = \frac{\sqrt{100^2+x^2}}{5} + \frac{\sqrt{100^2+(a-x)^2}}{2}$$

As $x \to \pm\infty$, $T \to \infty$, so the minimum value will be at a critical point.

$$T' = \frac{x}{5\sqrt{100^2+x^2}} - \frac{(a-x)}{2\sqrt{100^2+(a-x)^2}} = \frac{\sin\alpha}{5} - \frac{\sin\beta}{2}.$$

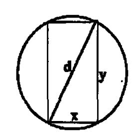
Therefore, if $T' = 0$, it follows that

$$\frac{\sin\alpha}{5} = \frac{\sin\beta}{2} \qquad \text{or} \qquad \frac{\sin\alpha}{\sin\beta} = \frac{5}{2}.$$

2C-11 Use implicit differentiation:

$x^2 + y^2 = d^2 \implies 2x + 2yy' = 0 \implies y' = -x/y.$

We want to maximize xy^3. At the endpoints $x = 0$ and $y = 0$, the strength is zero, so there is a maximum at a critical point. Differentiating,

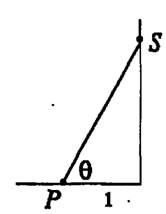


$$0 = (xy^3)' = y^3 + 3xy^2y' = y^3 + 3xy^2(-x/y) = y^3 - 3x^2y$$

Dividing by x^3,

$$(y/x)^3 - 3(y/x) = 0 \implies (y/x)^2 = 3 \implies y/x = \sqrt{3}.$$

2C-12 The intensity is proportional to



$$y = \frac{\sin\theta}{1+x^2} = \frac{x/\sqrt{1+x^2}}{1+x^2} = x(1+x^2)^{-3/2}$$

Endpoints: $y(0) = 0$ and $y \to 0$ as $x \to \infty$, so the maximum will be at a critical point. Critical points satisfy

$$y' = (1-2x^2)(1+x^2)^{-5/2} = 0 \implies 1 - 2x^2 = 0 \implies x = 1/\sqrt{2}$$

The best height is $1/\sqrt{2}$ feet above the desk. (It's not worth it. Use a desk lamp.)

2C-13 a) Let p denote the price in dollars. Then there will be $100 + (2/5)(200-p)$ passengers. Therefore the total revenue is

$$R = p(100 + (2/5)(200-p)) = p(180 - (2/5)p)$$

At the "ends" zero price $p = 0$, and no passengers $p = (5/2)180 = 450$, the revenue is zero. So the maximum occurs in between at a critical point.

$$R' = (180 - (2/5)p) - (2/5)p = 180 - (4/5)p = 0 \implies p = (5/4)180 = \$225$$

b)

$$P = xp - x(10 - x/10^5) \text{ with } x = 10^5(10 - p/2)$$

Therefore, the profit is cents is

$$P = 10^5(10 - p/2)(p - 10 + (10 - p/2)) = 10^5(10 - p/2)(p/2) = (10^5/4)p(20 - p)$$

$$\frac{dP}{dp} = (10^5/2)(10 - p)$$

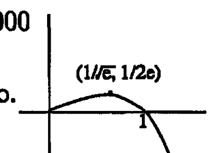
The critical point at $p = 10$. This is $x = 10^5(10 - 5) = 5 \times 10^5$ kilowatt hours, which is within the range available to the utility company. The function P has second derivative $-10^5/2$, so it is concave down and the critical point must be the maximum. (This is one of those cases where checking the second derivative is easier than checking the endpoints.)

Alternatively, the endpoint values are:

$$x = 0 \implies 10^5(10 - p/2) = 0 \implies p = 20 \implies P = 0.$$

$$\begin{aligned} x = 8 \times 10^5 &\implies 8 \times 10^5 = 10^5(10 - p/2) \\ &\implies 10 - p/2 = 8 \implies p = 4 \\ &\implies P = (10^5/4)4(20 - 4) = 16 \times 10^5 \text{cents} = \$160,000 \end{aligned}$$

The profit at the crit. pt. was $(10^5/4)10(20 - 10) = 2.5 \times 10^6$cents $= \$250,000$



14a

2C-14 a) Endpoints: $y = -x^2\ln(x) \to 0$ as $x \to 0^+$ and $y \to -\infty$ as $x \to \infty$.

Critical points: $y' = -2x\ln x - x = 0 \implies \ln x = -1/2 \implies x = 1/\sqrt{e}$.

Critical value: $y(1/\sqrt{e}) = 1/2e$.

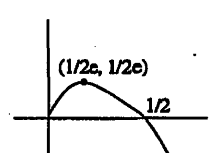
Maximum value: $1/2e$, attained when $x = 1/\sqrt{e}$. (min is not attained)

b) Endpoints: $y = -x\ln(2x) \to 0$ as $x \to 0^+$ and $y \to -\infty$ as $x \to \infty$.

Critical points: $y' = -\ln(2x) - 1 = 0 \implies x = 1/2e$.

Critical value: $y(1/2e) = -(1/2e)\ln(1/e) = 1/2e$.

Maximum value: $1/2e$, attained at $x = 1/2e$. (min is not attained)



14b

2C-15 No minimum. The derivative is $-xe^{-x} < 0$, so the function decreases. (Not needed here, but it will follows from E13/7 or from L'Hospital's rule in E31 that $xe^{-x} \to 0$ as $x \to \infty$.)

2D. More Max-min Problems

2D-3 The milk will be added at some time t_1, such that $0 \le t_1 \le 10$. In the interval $0 \le t < t_1$ the temperature is

$$y(t) = (100 - 20)e^{-(t-0)/10} + 20 = 80e^{-t/10} + 20$$

Therefore,

$$T_1 = y(t_1^-) = 80e^{-t_1/10} + 20$$

We are adding milk at a temperature $T_2 = 5$, so the temperature as we start the second interval of cooling is

$$\frac{9}{10}T_1 + \frac{1}{10}T_2 = 72e^{-t_1/10} + 18 + \frac{1}{2}$$

Let $Y(t)$ be the coffee temperature in the interval $t_1 \le t \le 10$. We have just calculated $Y(t_1)$, so

$$Y(t) = (Y(t_1) - 20)e^{-(t-t_1)/10} + 20 = (72e^{-t_1/10} - 1.5)e^{-(t-t_1)/10} + 20$$

The final temperature is

$$T = Y(10) = (72e^{-t_1/10} - 1.5)e^{-(10-t_1)/10} + 20 = e^{-1}\left(72 - (1.5)e^{t_1/10}\right) + 20$$

We want to maximize this temperature, so we look for critical points:

$$\frac{dT}{dt_1} = -(1.5/10e)e^{t_1/10} < 0$$

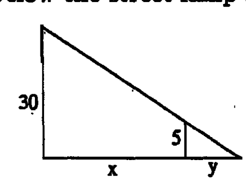
Therefore the function $T(t_1)$ is decreasing and its maximum occurs at the left endpoint: $t_1 = 0$.

Conclusion: The coffee will be hottest if you put the milk in as soon as possible.

2E. Related Rates

2E-1 The distance from robot to the point on the ground directly below the street lamp is $x = 20t$. Therefore, $x' = 20$.

$$\frac{x+y}{30} = \frac{y}{5} \quad \text{(similar triangles)}$$



Therefore,

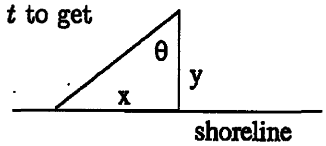
$$(x' + y')/30 = y'/5 \implies y' = 4 \text{ and } (x+y)' = 24$$

The tip of the shadow is moving at 24 feet per second and the length of the shadow is increasing at 4 feet per second.

2E-2

$$\tan\theta = \frac{x}{4} \quad \text{and } d\theta/dt = 3(2\pi) = 6\pi$$

with t is measured in minutes and θ measured in radians. The light makes an angle of 60° with the shore when θ is 30° or $\theta = \pi/6$. Differentiate with respect to t to get

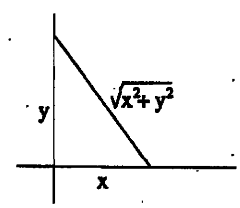


$$(\sec^2\theta)(d\theta/dt) = (1/4)(dx/dt)$$

Since $\sec^2(\pi/6) = 4/3$, we get $dx/dt = 32\pi$ miles per minute.

2E-3 The distance is $x = 10$, $y = 15$, $x' = 30$ and $y' = 30$. Therefore,

$$\begin{aligned}\left((x^2+y^2)^{1/2}\right)' &= (1/2)(2xx' + 2yy')(x^2+y^2)^{-1/2}\\ &= (10(30) + 15(30))/\sqrt{10^2+15^2}\\ &= 150/\sqrt{13}\text{miles per hour}\end{aligned}$$



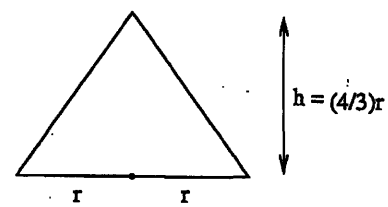
2E-4 $V = (\pi/3)r^2h$ and $2r = d = (3/2)h$ implies $h = (4/3)r$.

Therefore,

$$V = (\pi/3)r^2h = (4\pi/9)r^3$$

Moreover, $dV/dt = 12$, hence

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = \left(\frac{d}{dr}(4\pi/9)r^3\right)\frac{dr}{dt} = (4\pi/3)r^2(12) = 16\pi r^2.$$



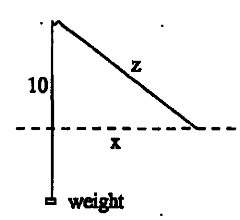
When $h = 2$, $r = 3/2$, so that $\dfrac{dV}{dt} = 36\pi\text{m}^3/\text{minute}$.

2E-5 The information is

$$x^2 + 10^2 = z^2, \quad z' = 4$$

We want to evaluate x' at $x = 20$. (Derivatives are with respect to time.) Thus

$$2xx' = 2zz' \text{ and } z^2 = 20^2 + 10^2 = 500$$

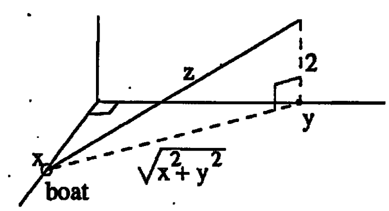


Therefore,

$$x' = (zz')/x = 4\sqrt{500}/20 = 2\sqrt{5}$$

2E-6 $x' = 50$ and $y' = 400$ and

$$z^2 = x^2 + y^2 + 2^2$$



The problem is to evaluate z' when $x = 50$ and $y = 400$. Thus

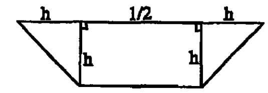
$$2zz' = 2xx' + 2yy' \implies z' = (xx' + yy')/z$$

and $z = \sqrt{50^2 + 400^2 + 4} = \sqrt{162504}$. So $z' = 162500/\sqrt{162504} \approx 403$mph.

(The fact that the plane is 2 miles up rather than at sea level changes the answer by only about 4/1000. Even the boat speed only affects the answer by about 3 miles per hour.)

2E-7 $V = 4(h^2 + h/2)$, $V' = 1$. To evaluate h' at $h = 1/2$,

$$1 = V' = 8hh' + 2h' = 8(1/2)h' + 2h' = 6h'$$

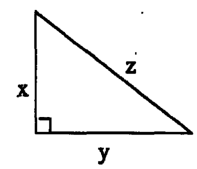


Cross-sectional area = h 2 + h/2

Therefore,

$$h' = 1/6 \text{ meters per second}$$

2E-8 $x' = 60$, $y' = 50$ and $x = 60 + 60t$, $y = 50t$. Noon is $t = 0$ and t is measured in hours. To find the time when $z = \sqrt{x^2 + y^2}$ is smallest, we may as well minimize $z^2 = x^2 + y^2$. We know that there will be a minimum at a critical point because when $t \to \pm\infty$ the distance tends to infinity. Taking the derivative with respect to t, the critical points satisfy



$$2xx' + 2yy' = 0$$

This equation says

$$2((60 + 60t)60 + (50t)50) = 0 \implies (60^2 + 50^2)t = -60^2$$

Hence

$$t = -36/61 \approx -35\text{min}$$

The ships were closest at around 11 : 25 am.

2E-9 $dy/dt = 2(x - 1)dx/dt$. Notice that in the range $x < 1$, $x - 1$ is negative and so $(x - 1) = -\sqrt{y}$. Therefore,

$$dx/dt = (1/2(x-1))(dy/dt) = -(1/2\sqrt{y})(dy/dt) = +(\sqrt{y})(1-y)/2 = 1/4\sqrt{2}$$

Method 2: Doing this directly turns out to be faster:

$$x = 1 - \sqrt{y} \implies dx/dt = 1 - (1/2)y^{-1/2}dy/dt$$

and the rest is as before.

2E-10 $r = Ct^{1/2}$. The implicit assumption is that the volume of oil is constant:

$$\pi r^2 T = V \text{ or } r^2 T = (V/\pi) = \text{const}$$

Therefore, differentiating with respect to time t,

$$(r^2T)' = 2rr'T + r^2T' = 0 \implies T' = -2r'/r$$

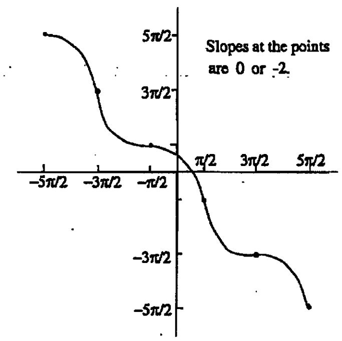
But $r' = (1/2)Ct^{-1/2}$, so that $r'/r = 1/2t$. Therefore

$$T' = -1/t$$

(Although we only know the rate of change of r up to a constant of proportionality, we can compute the absolute rate of change of T.)

2F. Locating zeros; Newton's method

2F-1 a) $y' = -\sin x - 1 \le 0$. Also, $y' < 0$ except at a discrete list of points (where $\sin x = -1$). Therefore y is strictly decreasing, that is, $x_1 < x_2 \implies y(x_1) < y(x_2)$. Thus y crosses zero only once.



Upper and lower bounds for z such that $y(z) = 0$:

$y(0) = 1$ and $y(\pi/2) = -\pi/2$. Therefore, $0 < z < \pi/2$.

b) $x_{n+1} = x_n - (\cos x_n - x_n)/(\sin x_n + 1)$

$$x_1 = 1, \qquad x_2 = .750363868,$$
$$x_3 = .739112891, \quad x_4 = .739085133$$

Accurate to three decimals at x_3, the second step. Answer .739.

c) Fixed point method takes 53 steps to stabilize at .739085133. Newton's method takes only three steps to get to 9 digits of accuracy. (See x_4.)

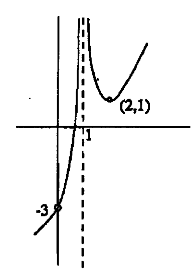
2F-2

$$y = 2x - 4 + \frac{1}{(x-1)^2} \qquad -\infty < x < \infty$$

$$y' = 2 - \frac{2}{(x-1)^3} = \frac{2((x-1)^3 - 1)}{(x-1)^3}$$

$y' = 0$ implies $(x-1)^3 = 1$, which implies $x - 1 = 1$ and hence that $x = 2$. The sign changes of y' are at the critical point $x = 2$ and at the singularity $x = 1$. For $x < 1$, the numerator and denominator are negative, so $y' > 0$. For $1 < x < 2$, the numerator is still negative, but the denominator is positive, so $y' < 0$. For $2 < x$, both numerator and denominator are positive, so $y' > 0$.

$$y'' = \frac{6}{(x-1)^4}$$



Therefore, $y'' > 0$ for $x \neq 1$.

Critical value: $y(2) = 1$

Singular values: $y(1^-) = y(1^+) = \infty$

Endpoint values: $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$.

Conclusion: The function increases from $-\infty$ to ∞ on the interval $(-\infty, 1)$. Therefore, the function vanishes exactly once in this interval. The function decreases from ∞ to 1 on the interval $(1, 2)$ and increases from 1 to ∞ on the interval $(2, \infty)$. Therefore, the function does not vanish at all in the interval $(1, \infty)$. Finally, the function is concave up on the intervals $(-\infty, 1)$ and $(1, \infty)$

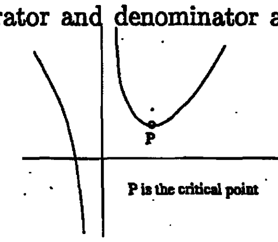
2F-3

$$y' = 2x - x^{-2} = \frac{2x^3 - 1}{x^2}$$

Therefore $y' = 0$ implies $x^3 = 1/2$ or $x = 2^{-1/3}$. Moreover, $y' > 0$ when $x > 2^{-1/3}$, and $y' < 0$ when $x < 2^{-1/3}$ and $x \neq 0$. The sign does not change across the singular point $x = 0$ because the power in the denominator is even. *(continued →)*

$$y'' = 2 + 2x^{-3} = \frac{2(x^3 + 1)}{x^3}$$

Therefore $y'' = 0$ implies $x^3 = -1$, or $x = -1$. Keeping track of the sign change in the denominator as well as the numerator we have that $y'' > 0$ when $x > 0$ and $y'' < 0$ when $-1 < x < 0$. Finally, $y'' > 0$ when $x < -1$, and both numerator and denominator are negative.



Critical value: $y(2^{-1/3}) = 2^{-2/3} + 2^{1/3} \approx 1.9$

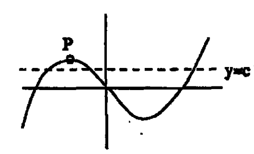
Singular value: $y(0^+) = +\infty$ and $y(0^-) = -\infty$

Endpoint values: $y \to \infty$ as $x \to \pm\infty$

Conclusions: The function decreases from ∞ to $-\infty$ in the interval $(-\infty, 0)$. Therefore it vanishes exactly once in this interval. It jumps to ∞ at 0 and decreases from ∞ to $2^{-2/3} + 2^{1/3}$ in the interval $(0, 2^{-1/3})$. Finally it increases from $2^{-2/3} + 2^{1/3}$ to ∞ in the interval $(2^{-1/3}, \infty)$. Thus it does not vanish on the interval $(0, \infty)$. The function is concave up in the intervals $(-\infty, -1)$ and $(0, \infty)$ and concave down in the interval $(-1, 0)$, with an inflection point at -1.

2F-4 From the graph, $x^5 - x - c = 0$ has three roots for any small value of c. The value of c gets too large if it exceeds the local maximum of $x^5 - x$ labelled. To calculate that local maximum, consider $y' = 5x^4 - 1 = 0$, with solutions $x = \pm 5^{-1/4}$. The local maximum is at $x = -5^{-1/4}$ and the value is

$$(-5^{-1/4})^5 - (-5^{-1/4}) = 5^{-1/4} - 5^{-5/4} \approx .535$$



Since $.535 > 1/2$, there are three roots.

2F-5 a) Answer: $x_1 = \pm 1/\sqrt{3}$. $f(x) = x - x^3$, so $f'(x) = 1 - 3x^2$ and

$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$

So x_2 is undefined if $f'(x_1) = 0$, that is $x_1 = \pm 1/\sqrt{3}$.

b) Answer: $x_1 = \pm 1/\sqrt{5}$. (This value can be found by experimentation. It can be also be found by iterating the inverse of the Newton method function.)

Here is an explanation: Using the fact that f is odd and that $x_3 = x_1$ suggests that $x_2 = -x_1$. This greatly simplifies the equation.

$$x_{n+1} = x_n - (x_n - x_n^3)/(1 - 3x_n^2) = \frac{-2x_n^3}{1 - 3x_n^2}$$

Therefore we want to find x satisfying

$$-x = \frac{-2x^3}{1-3x^2}$$

This equation is the same as $x(1-3x^2) = 2x^3$, which implies $x = 0$ or $5x^2 = 1$. In other words, $x = \pm 1/\sqrt{5}$. Now one can check that if $x_1 = 1/\sqrt{5}$, then $x_2 = -1/\sqrt{5}$, $x_3 = +1/\sqrt{5}$, etc.

c) Answers: If $x_1 < -1/\sqrt{3}$, then $x_n \to -1$. If $x_1 > 1/\sqrt{3}$, then $x_n \to 1$. If $-1/\sqrt{5} < x_1 < 1/\sqrt{5}$, then $x_n \to 0$. This can be found experimentally, numerically. For a complete analysis and proof one needs the methods of an upper level course like 18.100.

2F-6 a) To simplify this problem to its essence, let $V = \pi$. (We are looking for ratio r/h and this will be the same no matter what value we pick for V.) Thus $r^2h = 1$ and

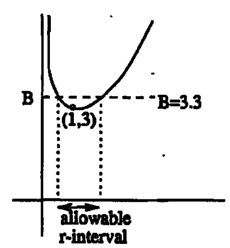
$$A = \pi r^2 + 2\pi/r$$

Minimize $B = A/\pi$ instead.

$$B = r^2 + 2r^{-1} \implies B' = 2r - 2r^{-2}$$

and $B' = 0$ implies $r = 1$. Endpoints: $B \to \infty$ as $r \to 0$ and as $r \to \infty$, so we have found the minimum at $r = 1$. (The constraint $r^2h = 1$ shows that this minimum is achieved when $r = h = 1$. As a doublecheck, the fact that the minimum area is achieved for $r/h = 1$ follows from 2C-5; see part (b).)

The minimum of B is 3 attained at $r = 1$. Ten percent more than the minimum is 3.3, so we need to find all r such that



$$B(r) \le 3.3$$

Use Newton's method with $F(r) = B(r) - 3.3$. (It is unwise to start Newton's method at $r = 1$. Why?) The roots of F are approximately $r = 1.35$ and $r = .72$.

Since $r^2h = 1$, $h = 1/r^2$ and the ratio,

$$r/h = r^3$$

Compute $(1.35)^3 \approx 2.5$ and $(.72)^3 \approx .37$. Therefore, the proportions with at most 10 percent extra glass are approximately

$$.37 < r/h < 2.5$$

b) The connection with Problem 2C-5 is that the minimum area $r = h$ is not entirely obvious, and not just because we are dealing with glass beakers instead of tin cans. In E10/5 the area is fixed whereas here the volume is held fixed. But because one needs a larger surface area to hold a larger volume, maximizing volume with fixed area is the same problem as minimizing surface area with fixed volume. This is an important so-called duality principle often used in optimization problems. In Problem 2C-5 the answer was $r = h$, which is the proportion with minimum surface area as confirmed in part (a).

2F-7 Minimize the distance squared, $x^2 + y^2$. The critical points satisfy

$$2x + 2yy' = 0$$

The constraint $y = \cos x$ implies $y' = -\sin x$. Therefore,

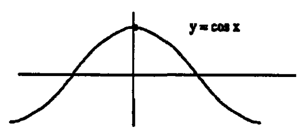
$$0 = x + yy' = x - \cos x \sin x$$

There is one obvious solution $x = 0$. The reason why this problem is in this section is that one needs the tools of inequalities to make sure that there are no other solutions. Indeed, if

$$f(x) = x - \cos x \sin x, \text{ then } f'(x) = 1 - \cos^2 x + \sin^2 x = 2\sin^2 x \geq 0$$

Furthermore, $f'(x)0$ is strictly positive except at the points $x = k\pi$, so f is increasing and crosses zero exactly once.

There is only one critical point and the distance tends to infinity at the endpoints $x \to \pm\infty$, so this point is the minimum. The point on the graph closest to the origin is $(0, 1)$.

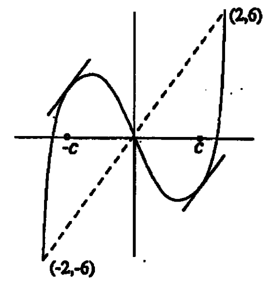


Alternative method: To show that $(0, 1)$ is closest it suffices to show that for $-1 \leq x < 0$ and $0 < x \leq 1$,

$$\sqrt{1 - x^2} < \cos x$$

Squaring gives $1 - x^2 < \cos^2 x$. This can be proved using the principles of problems 6 and 7. The derivative of $\cos^2 x - (1 - x^2)$ is twice the function f above, so the methods are very similar.

2G. Mean-value Theorem



2G-1 a) slope chord $= 1$; $f'(x) = 2x \Rightarrow f'(c) = 1$ if $c = \frac{1}{2}$.

b) slope chord $= \ln 2$; $f'(x) = \frac{1}{x} \Rightarrow f'(c) = \ln 2$ if $c = \frac{1}{\ln 2}$.

c) for $x^3 - x$: slope chord $= \dfrac{f(2) - f(-2)}{2-(-2)} = \dfrac{6-(-6)}{4} = 3$;

$f'(x) = 3x^2 - 1 \Rightarrow f'(c) = 3c^2 - 1 = 3 \Rightarrow c = \pm\frac{2}{\sqrt{3}}$

From the graph, it is clear you should get two values for c. (The axes are not drawn to the same scale.)

2G-2 a) $f(x) = f(a) + f'(c)(x-a)$; Take $a = 0$; $f(\xi) = \sin x$, $f'(x) = \cos x$

$\Rightarrow f(x) = 0 + \cos c \cdot x \Rightarrow \sin x < x$ (since $\cos c < 1$ for $0 < c < 2\pi$)

Thus the inequality is valid for $0 < x \le 2\pi$; since the function is periodic, it is also valid for all $x > 0$.

b) $\dfrac{d}{dx}\sqrt{1+x} = \dfrac{1}{2\sqrt{1+x}} \Rightarrow \sqrt{1+x} = 1 + \dfrac{1}{2\sqrt{1+c}}x < 1 + \dfrac{1}{2}x$, since $c > 0$.

2G-3. Let $s(t) =$ distance; then average velocity = slope of chord $= \dfrac{121}{11/6} = 66$.

Therefore, by MVT, there is some time $t = c$ such that $s'(c) = 66 > 65$.

(An application of the mean-value theorem to traffic enforcement...)

2G-4 According to Rolle's Theorem (Thm.1 p.800: an important special case of the M.V.T, and a step in its proof), between two roots of $p(x)$ lies at least one root of $p'(x)$. Therefore, between the n roots $a_1, \dots, a_n$ of $p(x)$, lie at least $n-1$ roots of $p'(x)$.

There are no more than $n-1$ roots, since degree of $p'(x) = n-1$; thus $p'(x)$ has exactly $n-1$ roots.

2G-5 Assume $f(x) = 0$ at a, b, c.

By Rolle's theorem (as in MVT-4), there are two points q_1, q_2 where $f'(q_1) = 0, f'(q_2) = 0$.

By Rolle's theorem again, applied to q_1 and q_2 and $f'(x)$, there is a point p where $f''(p) = 0$. Since p is between q_1 and q_2 , it is also between a and c.

2G-6 a) Given two points x_i such that $a \le x_1 < x_2 \le b$, we have

$$f(x_2) = f(x_1) + f'(c)(x_2 - x_1), \quad \text{where } x_1 < c < x_2.$$

Since $f'(x) > 0$ on $[a, b]$, $f'(c) > 0$; also $x_2 - x_1 > 0$. Therefore $f(x_2) > f(x_1)$, which shows $f(x)$ is increasing .

b) We have $f(x) = f(a) + f'(c)(x-a)$ where $a < c < x$.

Since $f'(c) = 0$, $f(x) = f(a)$ for $a \le x \le b$, which shows $f(x)$ is constant on $[a, b]$.

Unit 3. Integration

3A. Differentials, indefinite integration

3A-1 a) $7x^6dx$. ($d(\sin 1) = 0$ because $\sin 1$ is a constant.)

b) $(1/2)x^{-1/2}dx$

c) $(10x^9 - 8)dx$

d) $(3e^{3x}\sin x + e^{3x}\cos x)dx$

e) $(1/2\sqrt{x})dx + (1/2\sqrt{y})dy = 0$ implies

$$dy = -\frac{1/2\sqrt{x}dx}{1/2\sqrt{y}} = -\frac{\sqrt{y}}{\sqrt{x}}dx = -\frac{1-\sqrt{x}}{\sqrt{x}}dx = \left(1 - \frac{1}{\sqrt{x}}\right)dx$$

3A-2 a) $(2/5)x^5 + x^3 + x^2/2 + 8x + c$

b) $(2/3)x^{3/2} + 2x^{1/2} + c$

c) Method 1 (slow way) Substitute: $u = 8 + 9x$, $du = 9dx$. Therefore

$$\int \sqrt{8+9x}dx = \int u^{1/2}(1/9)du = (1/9)(2/3)u^{3/2} + c = (2/27)(8+9x)^{3/2} + c$$

Method 2 (guess and check): It's often faster to guess the form of the antiderivative and work out the constant factor afterwards:

$$\text{Guess } (8+9x)^{3/2}; \quad \frac{d}{dx}(8+9x)^{3/2} = (3/2)(9)(8+9x)^{1/2} = \frac{27}{2}(8+9x)^{1/2}.$$

So multiply the guess by $\frac{2}{27}$ to make the derivative come out right; the answer is then

$$\frac{2}{27}(8+9x)^{3/2} + c$$

d) Method 1 (slow way) Use the substitution: $u = 1 - 12x^4$, $du = -48x^3dx$.

$$\int x^3(1-12x^4)^{1/8}dx = \int u^{1/8}(-1/48)du = -\frac{1}{48}(8/9)u^{9/8} + c = -\frac{1}{54}(1-12x^4)^{9/8} + c$$

Method 2 (guess and check): guess $(1-12x^4)^{9/8}$;

$$\frac{d}{dx}(1-12x^4)^{9/8} = \frac{9}{8}(-48x^3)(1-12x^4)^{1/8} = -54(1-12x^4)^{1/8}.$$

So multiply the guess by $-\frac{1}{54}$ to make the derivative come out right, getting the previous answer.

e) Method 1 (slow way): Use substitution: $u = 8 - 2x^2$, $du = -4xdx$.

$$\int \frac{x}{\sqrt{8-2x^2}}dx = \int u^{1/2}(-1/4)du = -\frac{1}{4}\frac{2}{3}u^{3/2} + c = -\frac{1}{6}(8-2x^2)^{3/2} + c$$

Method 2 (guess and check): guess $(8-2x^2)^{3/2}$; differentiating it:

$$\frac{d}{dx}(8-2x^2)^{3/2} = \frac{3}{2}(-4x^2)(8-2x^2)^{1/2} = -6(8-2x^2)^{1/2};$$

so multiply the guess by $-\dfrac{1}{6}$ to make the derivative come out right.

The next four questions you should try to do (by Method 2) in your head. Write down the correct form of the solution and correct the factor in front.

f) $(1/7)e^{7x}+c$

g) $(7/5)e^{x^5}+c$

h) $2e^{\sqrt{x}}+c$

i) $(1/3)\ln(3x+2)+c$. For comparison, let's see how much slower substitution is:

$$u = 3x+2, \quad du = 3dx, \quad \text{so}$$

$$\int \frac{dx}{3x+2} = \int \frac{(1/3)du}{u} = (1/3)\ln u + c = (1/3)\ln(3x+2)+c$$

j)

$$\int \frac{x+5}{x}dx = \int \left(1+\frac{5}{x}\right)dx = x + 5\ln x + c$$

k)

$$\int \frac{x}{x+5}dx = \int \left(1-\frac{5}{x+5}\right)dx = x - 5\ln(x+5)+c$$

In Unit 5 this sort of algebraic trick will be explained in detail as part of a general method. What underlies the algebra in both (j) and (k) is the algorithm of long division for polynomials.

l) $u = \ln x$, $du = dx/x$, so

$$\int \frac{\ln x}{x}dx = \int u\,du = (1/2)u^2 + c = (1/2)(\ln x)^2 + c$$

m) $u = \ln x$, $du = dx/x$.

$$\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln u + c = \ln(\ln x) + c$$

3A-3 a) $-(1/5)\cos(5x)+c$

b) $(1/2)\sin^2 x + c$, coming from the substitution $u = \sin x$ or $-(1/2)\cos^2 x + c$, coming from the substitution $u = \cos x$. The two functions $(1/2)\sin^2 x$ and $-(1/2)\cos^2 x$ are not the same. Nevertheless the two answers given are the same. Why? (See 1J-1(m).)

c) $-(1/3)\cos^3 x + c$

d) $-(1/2)(\sin x)^{-2} + c = -(1/2)\csc^2 x + c$

e) $5\tan(x/5) + c$

f) $(1/7)\tan^7 x + c$.

g) $u = \sec x$, $du = \sec x \tan x dx$,

$$\int \sec^9 x \tan x dx \int (\sec x)^8 \sec x \tan x dx = (1/9)\sec^9 x + c$$

3B. Definite Integrals

3B-1 a) $1 + 4 + 9 + 16 = 30$ b) $2 + 4 + 8 + 16 + 32 + 64 = 126$

c) $-1 + 4 - 9 + 16 - 25 = -15$ d) $1 + 1/2 + 1/3 + 1/4 = 25/12$

3B-2 a) $\sum_{n=1}^{6}(-1)^{n+1}(2n+1)$ b) $\sum_{k=1}^{n} 1/k^2$ c) $\sum_{k=1}^{n}\sin(kx/n)$

3B-3 a) upper sum = right sum = $(1/4)[(1/4)^3 + (2/4)^3 + (3/4)^3 + (4/4)^3] = 15/128$

lower sum = left sum = $(1/4)[0^3 + (1/4)^3 + (2/4)^3 + (3/4)^3] = 7/128$

b) left sum $= (-1)^2 + 0^2 + 1^2 + 2^2 = 6$; right sum $= 0^2 + 1^2 + 2^2 + 3^2 = 14$;

upper sum $= (-1)^2 + 1^2 + 2^2 + 3^2 = 15$; lower sum $= 0^2 + 0^2 + 1^2 + 2^2 = 5$.

c) left sum $= (\pi/2)[\sin 0 + \sin(\pi/2) + \sin(\pi) + \sin(3\pi/2)] = (\pi/2)[0 + 1 + 0 - 1] = 0$;

right sum $= (\pi/2)[\sin(\pi/2) + \sin(\pi) + \sin(3\pi/2) + \sin(2\pi)] = (\pi/2)[1 + 0 - 1 + 0] = 0$;

upper sum $= (\pi/2)[\sin(\pi/2) + \sin(\pi/2) + \sin(\pi) + \sin(2\pi)] = (\pi/2)[1 + 1 + 0 + 0] = \pi$;

lower sum $= (\pi/2)[\sin(0) + \sin(\pi) + \sin(3\pi/2) + \sin(3\pi/2)] = (\pi/2)[0 + 0 - 1 - 1] = -\pi$.

3B-4 Both x^2 and x^3 are increasing functions on $0 \le x \le b$, so the upper sum is the right sum and the lower sum is the left sum. The difference between the right and left Riemann sums is

$$(b/n)[f(x_1 + \cdots + f(x_n)] - (b/n)[f(x_0 + \cdots + f(x_{n-1})] = (b/n)[f(x_n) - f(x_0)]$$

In both cases $x_n = b$ and $x_0 = 0$, so the formula is

$$(b/n)(f(b) - f(0))$$

a) $(b/n)(b^2 - 0) = b^3/n$. Yes, this tends to zero as $n \to \infty$.

b) $(b/n)(b^3 - 0) = b^4/n$. Yes, this tends to zero as $n \to \infty$.

3B-5 The expression is the right Riemann sum for the integral

$$\int_0^1 \sin(bx)dx = -(1/b)\cos(bx)|_0^1 = (1 - \cos b)/b$$

so this is the limit.

3C. Fundamental theorem of calculus

3C-1

$$\int_3^6 (x-2)^{-1/2}dx = 2(x-2)^{1/2}\Big|_3^6 = 2[(4)^{1/2} - 1^{1/2}] = 2$$

3C-2 a) $(2/3)(1/3)(3x+5)^{3/2}\Big|_0^2 = (2/9)(11^{3/2} - 5^{3/2})$

b) If $n \neq -1$, then

$$(1/(n+1))(1/3)(3x+5)^{n+1}\Big|_0^2 = (1/3(n+1))((11^{n+1} - 5^{n+1})$$

If $n = -1$, then the answer is $(1/3)\ln(11/5)$.

c) $(1/2)(\cos x)^{-2}\Big|_{3\pi/4}^{\pi} = (1/2)[(-1)^{-2} - (-1/\sqrt{2})^{-2}] = -1/2$

3C-3 a) $(1/2)\ln(x^2+1)\Big|_1^2 = (1/2)[\ln 5 - \ln 2] = (1/2)\ln(5/2)$

b) $(1/2)\ln(x^2+b^2)\Big|_b^{2b} = (1/2)[\ln(5b^2) - \ln(2b^2)] = (1/2)\ln(5/2)$

3C-4 As $b \to \infty$,

$$\int_1^b x^{-10}dx = -(1/9)x^{-9}\Big|_1^b = -(1/9)(b^{-9}-1) \to -(1/9)(0-1) = 1/9.$$

This integral is the area of the infinite region between the curve $y = x^{-10}$ and the x-axis for $x > 0$.

3C-5 a) $\displaystyle\int_0^{\pi} \sin x dx = -\cos x\Big|_0^{\pi} = -(\cos\pi - \cos 0) = 2$

b) $\displaystyle\int_0^{\pi/a} \sin(ax)dx = -(1/a)\cos(ax)\Big|_0^{\pi/a} = -(1/a)(\cos\pi - \cos 0) = 2/a$

3C-6 a) $x^2 - 4 = 0$ implies $x = \pm 2$. So the area is

$$\int_{-2}^2 (x^2-4)dx = 2\int_0^2 (x^2-4)dx = \frac{x^3}{3} - 4x\Big|_0^2 = \frac{8}{3} - 4\cdot 2 = -16/3$$

(We changed to the interval $(0,2)$ and doubled the integral because x^2-4 is even.) Notice that the integral gave the wrong answer! It's negative. This is because the graph $y = x^2 - 4$ is concave up and is below the x-axis in the interval $-2 < x < 2$. So the correct answer is $16/3$.

b) Following part (a), $x^2 - a = 0$ implies $x = \pm\sqrt{a}$. The area is

$$\int_{-\sqrt{a}}^{\sqrt{a}} (a-x^2)dx = 2\int_0^{\sqrt{a}} (a-x^2)dx = 2ax - \frac{x^3}{3}\Big|_0^{\sqrt{a}} = 2(a^{3/2} - \frac{a^{3/2}}{3}) = \frac{4}{3}a^{3/2}$$

3D. Second fundamental theorem

3D-1 Differentiate both sides;

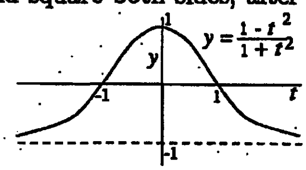
left side $L(x)$: $L'(x) = \frac{d}{dx}\int_0^x \frac{dt}{\sqrt{a^2+x^2}} = \frac{1}{\sqrt{a^2+x^2}}$, by FT2;

right side $R(x)$: $R'(x) = \frac{d}{dx}(\ln(x+\sqrt{a^2+x^2}) - \ln a) = \frac{1+\frac{x}{\sqrt{a^2+x^2}}}{x+\sqrt{a^2+x^2}} = \frac{1}{\sqrt{a^2+x^2}}$

Since $L'(x) = R'(x)$, we have $L(x) = R(x) + C$ for some constant $C = L(x) - R(x)$. The constant C may be evaluated by assigning a value to x; the most convenient choice is $x = 0$, which gives

$$L(0) = \int_0^0 = 0; \quad R(0) = \ln(0+\sqrt{0+a^2}) - \ln a = 0; \text{ therefore } C = 0 \text{ and } L(x) = R(x).$$

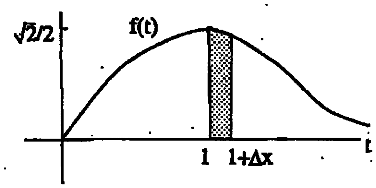
b) Put $x = c$; the equation becomes $0 = \ln(c+\sqrt{c^2+a^2})$; solve this for c by first exponentiating both sides: $1 = c+\sqrt{c^2+a^2}$; then subtract c and square both sides; after some algebra one gets $c = \frac{1}{2}(1-a^2)$.



3D-3 Sketch $y = \frac{1-t^2}{1+t^2}$ first, as shown at the right.

3D-4 a) $\int_0^x \sin(t^3)dt$, by the FT2. b) $\int_0^x \sin(t^3)dt + 2$ c) $\int_1^x \sin(t^3)dt - 1$

3D-5 This problem reviews the idea of the proof of the FT2.



a) $f(t) = \frac{t}{\sqrt{1+t^4}}$

$$\frac{1}{\Delta x}\int_1^{1+\Delta x} f(t)dt = \frac{\text{shaded area}}{\text{width}} \approx \text{height} .$$

$$\lim_{\Delta x\to 0} \frac{1}{\Delta x}\int_1^{1+\Delta x} f(t)dt = \lim_{\Delta x\to 0} \frac{\text{shaded area}}{\text{width}} = \text{height} = f(1) = \frac{1}{\sqrt{2}}.$$

b) By definition of derivative,

$$F'(1) = \lim_{\Delta x\to 0} \frac{F(1+\Delta x) - F(1)}{\Delta x} = \lim_{\Delta x\to 0} \frac{1}{\Delta x}\int_1^{1+\Delta x} f(t)dt;$$

by FT2, $F'(1) = f(1) = \frac{1}{\sqrt{2}}$.

3D-6 a) If $F_1(x) = \int_{a_1}^x dt$ and $F_2(x) = \int_{a_2}^x dt$, then $F_1(x) = x - a_1$ and $F_2(x) = x - a_2$. Thus $F_1(x) - F_2(x) = a_2 - a_1$, a constant.

b) By the FT2, $F_1'(x) = f(x)$ and $F_2'(x) = f(x)$; therefore $F_1 = F_2 + C$, for some constant C.

3D-7 a) Using the FT2 and the chain rule, as in the Notes,

$$\frac{d}{dx}\int_0^{x^2}\sqrt{u}\sin u\,du = \sqrt{x^2}\sin(x^2)\cdot\frac{d(x^2)}{dx} = 2x^2\sin(x^2)$$

$$\text{b)} = \frac{1}{\sqrt{1-\sin^2 x}}\cdot\cos x = 1. \quad \left(\text{So } \int_0^{\sin x}\frac{dt}{1-t^2} = x\right)$$

$$\text{c)}\ \frac{d}{dx}\int_x^{x^2}\tan u\,du = \tan(x^2)\cdot 2x - \tan x$$

3D-8 a) Differentiate both sides using FT2, and substitute $x = \pi/2$: $f(\pi/2) = 4$.

b) Substitute $x = 2u$ and follow the method of part (a); put $u = \pi$, get finally $f(\pi/2) = 4 - 4\pi$.

3E. Change of Variables; Estimating Integrals

3E-1 $L(\frac{1}{a}) = \int_1^{1/a}\frac{dt}{t}$. Put $t = \frac{1}{u}$, $dt = -\frac{1}{u^2}du$. Then

$$\frac{dt}{t} = -\frac{u}{u^2}du \implies L(\frac{1}{a}) = \int_1^{1/a}\frac{dt}{t} = -\int_1^a\frac{du}{u} = -L(a)$$

3E-2 a) We want $-t^2 = -u^2/2$, so $u = t\sqrt{2}$, $du = \sqrt{2}dt$.

$$\frac{1}{\sqrt{2\pi}}\int_0^x e^{-u^2/2}du = \frac{1}{\sqrt{2\pi}}\int_0^{x/\sqrt{2}} e^{-t^2}\sqrt{2}dt = \frac{1}{\sqrt{\pi}}\int_0^{x/\sqrt{2}} e^{-t^2}dt$$

$$\implies E(x) = \frac{1}{\sqrt{\pi}}F(x/\sqrt{2}) \quad \text{and} \quad \lim_{x\to\infty}E(x) = \frac{1}{\pi}\cdot\frac{\sqrt{\pi}}{2} = \frac{1}{2}$$

b) The integrand is even, so

$$\frac{1}{\sqrt{2\pi}}\int_{-N}^N e^{-u^2/2}du = \frac{2}{\sqrt{2\pi}}\int_0^N e^{-u^2/2}du = 2E(N) \longrightarrow 1 \quad \text{as } N\to\infty$$

$$\lim_{x\to-\infty}E(x) = -1/2 \quad \text{because } E(x) \text{ is odd.}$$

$$\frac{1}{\sqrt{2\pi}}\int_a^b e^{-u^2/2}du = E(b) - E(a) \quad \text{by FT1 or by "interval addition" Notes PI (3).}$$

Commentary: The answer is consistent with the limit,

$$\frac{1}{\sqrt{2\pi}}\int_{-N}^N e^{-u^2/2}du = E(N) - E(-N) = 2E(N) \longrightarrow 1 \text{ as } N\to\infty$$

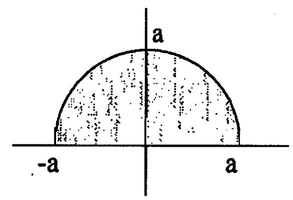
3E-3 a) Using $u = \ln x$, $du = \dfrac{dx}{x}$, $\displaystyle\int_1^e \frac{\sqrt{\ln x}}{x}dx = \int_0^1 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_0^1 = \frac{2}{3}$.

b) Using $u = \cos x$, $du = -\sin x$,

$$\int_0^{\pi} \frac{\sin x}{(2+\cos x)^3}dx = \int_1^{-1} \frac{-du}{(2+u)^3} = \frac{1}{2(2+u)^2}\Big|_1^{-1} = \frac{1}{2}\left(\frac{1}{1^2} - \frac{1}{3^2}\right) = \frac{4}{9}.$$

c) Using $x = \sin u$, $dx = \cos u\,du$, $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \int_0^{\frac{\pi}{2}} \frac{\cos u}{\cos u}du = u\Big|_0^{\pi/2} = \frac{\pi}{2}$.

3E-4 Substitute $x = t/a$; then $x = \pm 1 \Rightarrow t = \pm a$. We then have



$$\frac{\pi}{2} = \int_{-1}^{1} \sqrt{1-x^2}dx = \int_{-a}^{a} \sqrt{1 - \frac{t^2}{a^2}}\frac{dt}{a} = \frac{1}{a^2}\int_{-a}^{a} \sqrt{a^2 - t^2}dt.$$

Multiplying by a^2 gives the value $\pi a^2/2$ for the integral, which checks, since the integral represents the area of the semicircle.

3E-5 One can use informal reasoning based on areas (as in Ex. 5, Notes FT), but it is better to use change of variable.

a) Goal: $F(-x) = -F(x)$. Let $t = -u$, $dt = -du$, then

$$F(-x) = \int_0^{-x} f(t)dt = \int_0^{x} f(-u)(-du)$$

Since f is even ($f(-u) = f(u)$), $F(-x) = -\displaystyle\int_0^x f(u)du = -F(x)$.

b) Goal: $F(-x) = F(x)$. Let $t = -u$, $dt = -du$, then

$$F(-x) = \int_0^{-x} f(t)dt = \int_0^{x} f(-u)(-du)$$

Since f is odd (($f(-u) = -f(u)$), $F(-x) = \displaystyle\int_0^x f(u)du = F(x)$.

3E-6 a) $x^3 < x$ on (0,1) $\Rightarrow \dfrac{1}{1+x^3} > \dfrac{1}{1+x}$ on (0,1); therefore

$$\int_0^1 \frac{dx}{1+x^3} > \int_0^1 \frac{dx}{1+x} = \ln(1+x)\Big|_0^1 = \ln 2 = .69$$

b) $0 < \sin x < 1$ on $(0,\pi)$ $\Rightarrow \sin^2 x < \sin x$ on $(0,\pi)$; therefore

$$\int_0^{\pi} \sin^2 x dx < \int_0^{\pi} \sin x dx = -\cos x\Big|_0^{\pi} = -(-1-1) = 2.$$

c) $\displaystyle\int_{10}^{20} \sqrt{x^2+1}dx > \int_{10}^{20} \sqrt{x^2}dx = \frac{x^2}{2}\Big|_{10}^{20} = \frac{1}{2}(400-100) = 150$

3E-7 $\displaystyle\left|\int_1^N \frac{\sin x}{x^2}dx\right| \le \int_1^N \frac{|\sin x|}{x^2}dx \le \int_1^N \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^N = -\frac{1}{N} + 1 < 1.$

3F. Differential Equations: Separation of Variables. Applications

3F-1 a) $y = (1/10)(2x+5)^5 + c$

b) $(y+1)dy = dx \implies \int (y+1)dy = \int dx \implies (1/2)(y+1)^2 = x + c$. You can leave this in implicit form or solve for y: $y = -1 \pm \sqrt{2x+a}$ for any constant a $(a = 2c)$

c) $y^{1/2}dy = 3dx \implies (2/3)y^{3/2} = 3x + c \implies y = (9x/2 + a)^{2/3}$, with $a = (3/2)c$.

d) $y^{-2}dy = xdx \implies -y^{-1} = x^2/2 + c \implies y = -1/(x^2/2 + c)$

3F-2 a) Answer: $3e^{16}$.

$$y^{-1}dy = 4xdx \implies \ln y = 2x^2 + c$$

$$y(1) = 3 \implies \ln 3 = 2 + c \implies c = \ln 3 - 2.$$

Therefore

$$\ln y = 2x^2 + (\ln 3 - 2)$$

At $x = 3$, $y = e^{18+\ln 3 - 2} = 3e^{16}$

b) Answer: $y = 11/2 + 3\sqrt{2}$.

$$(y+1)^{-1/2}dy = dx \implies 2(y+1)^{1/2} = x + c$$

$$y(0) = 1 \implies 2(1+1)^{1/2} = c \implies c = 2\sqrt{2}$$

At $x = 3$,

$$2(y+1)^{1/2} = 3 + 2\sqrt{2} \implies y + 1 = (3/2 + \sqrt{2})^2 = 13/2 + 3\sqrt{2}$$

Thus, $y = 11/2 + 3\sqrt{2}$.

c) Answer: $y = \sqrt{550/3}$

$$ydy = x^2dx \implies y^2/2 = (1/3)x^3 + c$$

$$y(0) = 10 \implies c = 10^2/2 = 50$$

Therefore, at $x = 5$,

$$y^2/2 = (1/3)5^3 + 50 \implies y = \sqrt{550/3}$$

d) Answer: $y = (2/3)(e^{24} - 1)$

$$(3y+2)^{-1}dy = dx \implies (1/3)\ln(3y+2) = x + c$$

$$y(0) = 0 \implies (1/3)\ln 2 = c$$

Therefore, at $x = 8$,

$$(1/3)\ln(3y+2) = 8 + (1/3)\ln 2 \implies \ln(3y+2) = 24 + \ln 2 \implies (3y+2) = 2e^{24}$$

Therefore, $y = (2e^{24} - 2)/3$

e) Answer: $y = -\ln 4$ at $x = 0$. Defined for $-\infty < x < 4$.

$$e^{-y}dy = dx \implies -e^{-y} = x + c$$

$$y(3) = 0 \implies -e^{0} = 3 + c \implies c = -4$$

Therefore,

$$y = -\ln(4 - x), \quad y(0) = -\ln 4$$

The solution y is defined only if $x < 4$.

3F-3 a) Answers: $y(1/2) = 2$, $y(-1) = 1/2$, $y(1)$ is undefined.

$$y^{-2}dy = dx \implies -y^{-1} = x + c$$

$$y(0) = 1 \implies -1 = 0 + c \implies c = -1$$

Therefore, $-1/y = x - 1$ and

$$y = \frac{1}{1-x}$$

The values are $y(1/2) = 2$, $y(-1) = -1/2$ and y is undefined at $x = 1$.

b) Although the formula for y makes sense at $x = 3/2$, $(y(3/2) = 1/(1 - 3/2) = -2)$, it is not consistent with the rate of change interpretation of the differential equation. The function is defined, continuous and differentiable for $-\infty < x < 1$. But at $x = 1$, y and dy/dx are undefined. Since $y = 1/(1 - x)$ is the only solution to the differential equation in the interval $(0, 1)$ that satisfies the initial condition $y(0) = 1$, it is impossible to define a function that has the initial condition $y(0) = 1$ and also satisfies the differential equation in any longer interval containing $x = 1$.

To ask what happens to y after $x = 1$, say at $x = 3/2$, is something like asking what happened to a rocket ship after it fell into a black hole. There is no obvious reason why one has to choose the formula $y = 1/(1 - x)$ after the "explosion." For example, one could define $y = 1/(2 - x)$ for $1 \le x < 2$. In fact, any formula $y = 1/(c - x)$ for $c \ge 1$ satisfies the differential equation at every point $x > 1$.

3F-4 a) If the surrounding air is cooler $(T_e - T < 0)$, then the object will cool, so $dT/dt < 0$. Thus $k > 0$.

b) Separate variables and integrate.

$$(T - T_e)^{-1}dT = -kdt \implies \ln|T - T_e| = -kt + c$$

Exponentiating,

$$T - T_e = \pm e^{c}e^{-kt} = Ae^{-kt}$$

The initial condition $T(0) = T_0$ implies $A = T_0 - T_e$. Thus

$$T = T_e + (T_0 - T_e)e^{-kt}$$

c) Since $k > 0$, $e^{-kt} \to 0$ as $t \to \infty$. Therefore,

$$T = T_e + (T_0 - T_e)e^{-kt} \longrightarrow T_e \text{ as } t \to \infty$$

d)

$$T - T_e = (T_0 - T_e)e^{-kt}$$

The data are $T_0 = 680$, $T_e = 40$ and $T(8) = 200$. Therefore,

$$200 - 40 = (680 - 40)e^{-8k} \implies e^{-8k} = 160/640 = 1/4 \implies -8k = -\ln 4.$$

The number of hours t that it takes to cool to $50°$ satisfies the equation

$$50 - 40 = (640)e^{-kt} \implies e^{-kt} = 1/64 \implies -kt = -3\ln 4.$$

To solve the two equations on the right above simultaneously for t, it is easiest just to divide the bottom equation by the top equation, which gives

$$\frac{t}{8} = 3, \quad t = 24.$$

e)

$$T - T_e = (T_0 - T_e)e^{-kt}$$

The data at $t = 1$ and $t = 2$ are

$$800 - T_e = (1000 - T_e)e^{-k} \quad \text{and} \quad 700 - T_e = (1000 - T_e)e^{-2k}$$

Eliminating e^{-k} from these two equations gives

$$\begin{aligned}
\frac{700 - T_e}{1000 - T_e} &= \left(\frac{800 - T_e}{1000 - T_e}\right)^2 \\
(800 - T_e)^2 &= (1000 - T_e)(700 - T_e) \\
800^2 - 1600T_e + T_e^2 &= (1000)(700) - 1700T_e + T_e^2 \\
100T_e &= (1000)(700) - 800^2 \\
T_e &= 7000 - 6400 = 600
\end{aligned}$$

f) To confirm the differential equation:

$$y'(t) = T'(t - t_0) = k(T_e - T(t - t_0)) = k(T_e - y(t))$$

The formula for y is

$$y(t) = T(t - t_0) = T_e + (T_0 - T_e)e^{-k(t-t_0)} = a + (y(t_0) - a)e^{-c(t-t_0)}$$

with $k = c$, $T_e = a$ and $T_0 = T(0) = y(t_0)$.

3F-6 $y = \cos^3 u - 3\cos u$, $x = \sin^4 u$

$dy = (3\cos^2 u \cdot (-\sin u) + 3\sin u)du$, $dx = 4\sin^3 u \cos u du$

$$\frac{dy}{dx} = \frac{3\sin u(1 - \cos^2 u)}{4\sin^3 u \cos u} = \frac{3}{4\cos u}$$

3F-7 a) $y' = -xy;\ y(0) = 1$

$\frac{dy}{y} = -xdx \implies \ln y = -\frac{1}{2}x^2 + c$

To find c, put $x = 0$, $y = 1$: $\ln 1 = 0 + c \implies c = 0$.

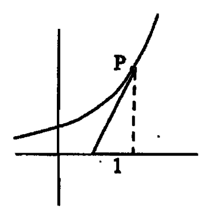
$\implies \ln y = -\frac{1}{2}x^2 \implies y = e^{-x^2/2}$

b) $\cos x \sin y dy = \sin x dx;\ y(0) = 0$

$\sin y dy = \frac{\sin x}{\cos x} dx \implies -\cos y = -\ln(\cos x) + c$

Find c: put $x = 0$, $y = 0$: $-\cos 0 = -\ln(\cos 0) + c \implies c = -1$

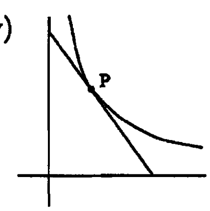
$\implies \cos y = \ln(\cos x) + 1$



3F-8 a) From the triangle, $y' = \text{slope tangent} = \frac{y}{1}$

$\implies \frac{dy}{y} = dx \implies \ln y = x + c_1 \implies y = e^{x+c_1} = Ae^x \ (A = e^{c_1})$

b) If P bisects tangent, then P_0 bisects OQ (by euclidean geometry)



So $P_0Q = x$ (since $OP_0 = x$).

Slope tangent $= y' = \frac{-y}{x} \implies \frac{dy}{y} = -\frac{dx}{x}$

$\implies \ln y = -\ln x + c_1$

Exponentiate: $y = \frac{1}{x} \cdot e^{c_1} = \frac{c}{x}, c > 0$

Ans: The hyperbolas $y = \frac{c}{x}, c > 0$

3G. Numerical Integration

3G-1 Left Riemann sum: $(\Delta x)(y_0 + y_1 + y_2 + y_3)$

Trapezoidal rule: $(\Delta x)((1/2)y_0 + y_1 + y_2 + y_3 + (1/2)y_4)$

Simpson's rule: $(\Delta x/3)(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$

a) $\Delta x = 1/4$ and

$$y_0 = 0,\ y_1 = 1/2,\ y_2 = 1/\sqrt{2},\ y_3 = \sqrt{3}/2,\ y_4 = 1.$$

Left Riemann sum: $(1/4)(0 + 1/2 + 1/\sqrt{2} + \sqrt{3}/2) \approx .518$

Trapezoidal rule: $(1/4)((1/2) \cdot 0 + 1/2 + 1/\sqrt{2} + \sqrt{3}/2 + (1/2)1) \approx .643$

Simpson's rule: $(1/12)(1 \cdot 0 + 4(1/2) + 2(1/\sqrt{2}) + 4(\sqrt{3}/2) + 1) \approx .657$

as compared to the exact answer .6666...

b) $\Delta x = \pi/4$

$$y_0 = 0,\ y_1 = 1/\sqrt{2},\ y_2 = 1,\ y_3 = 1/\sqrt{2},\ y_4 = 0.$$

Left Riemann sum: $(\pi/4)(0+1/\sqrt{2}+1+1/\sqrt{2}) \approx 1.896$

Trapezoidal rule: $(\pi/4)((1/2)\cdot 0+1/\sqrt{2}+1+1/\sqrt{2}+(1/2)\cdot 0) \approx 1.896$ (same as Riemann sum)

Simpson's rule: $(\pi/12)(1\cdot 0+4(1/\sqrt{2})+2(1)+4(1/\sqrt{2})+1\cdot 0) \approx 2.005$

as compared to the exact answer 2

c) $\Delta x = 1/4$

$$y_0 = 1,\ y_1 = 16/17,\ y_2 = 4/5,\ y_3 = 16/25,\ y_4 = 1/2.$$

Left Riemann sum: $(1/4)(1+16/17+4/5+16/25) \approx .845$

Trapezoidal rule: $(1/4)((1/2)\cdot 1+16/17+4/5+16/25+(1/2)(1/2)) \approx .8128$

Simpson's rule: $(1/12)(1\cdot 1+4(16/17)+2(4/5)+4(16/25)+1(1/2)) \approx .785392$

as compared to the exact answer $\pi/4 \approx .785398$

(Multiplying the Simpson's rule answer by 4 gives a passable approximation to π, of 3.14157, accurate to about 2×10^{-5}.)

d) $\Delta x = 1/4$

$$y_0 = 1,\ y_1 = 4/5,\ y_2 = 2/3,\ y_3 = 4/7,\ y_4 = 1/2.$$

Left Riemann sum: $(1/4)(1+4/5+2/3+4/7) \approx .76$

Trapezoidal rule: $(1/4)((1/2)\cdot 1+4/5+2/3+4/7(1/2)(1/2)) \approx .697$

Simpson's rule: $(1/12)(1\cdot 1+4(4/5)+2(2/3)+4(4/7)+1(1/2)) \approx .69325$

Compared with the exact answer $\ln 2 \approx .69315$, Simpson's rule is accurate to about 10^{-4}.

3G-2 We have $\displaystyle\int_0^b x^3dx = \frac{b^4}{4}$. Using Simpson's rule with two subintervals, $\Delta x = b/2$, so that we get the same answer as above:

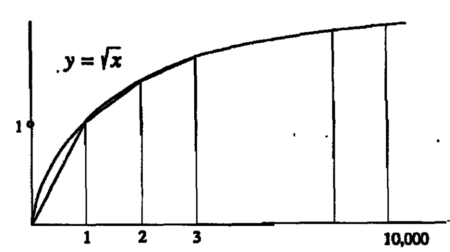
$$S(x^3) = \frac{b}{6}(0+4(b/2)^3+b^3) = \frac{b}{6}\left(\frac{3}{2}b^3\right) = \frac{b^4}{4}.$$

Remark. The fact that Simpson's rule is exact on cubic polynomials is very significant to its effectiveness as a numerical approximation. It implies that the approximation converges at a rate proportional to the the fourth derivative of the function times $(\Delta x)^4$, which is fast enough for many practical purposes.

3G-3 The sum

$$S = \sqrt{1}+\sqrt{2}+\dots+\sqrt{10,000}$$

is related to the trapezoidal estimate of $\displaystyle\int_0^{10^4}\sqrt{x}dx$:



(1) $$\int_0^{10^4} \sqrt{x}dx \approx \frac{1}{2}\sqrt{0}+\sqrt{1}+...+\frac{1}{2}\sqrt{10^4} = S - \frac{1}{2}\sqrt{10^4}$$

But

$$\int_0^{10^4} \sqrt{x}dx = \frac{2}{3}x^{3/2}\Big|_0^{10^4} = \frac{2}{3}\cdot 10^6$$

From (1),

(2) $$\frac{2}{3}\cdot 10^6 \approx S - 50$$

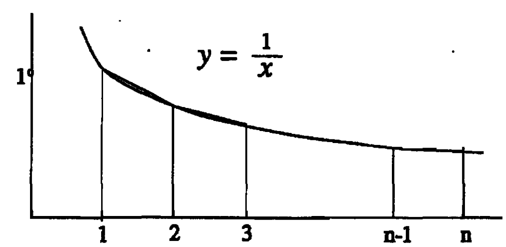
Hence

(3) $$S \approx 666,717$$

In (1), we have >, as in the picture. Hence in (2), we have >, so in (3), we have <, Too high.

3G-4 As in Problem 3 above, let

$$S = \frac{1}{1}+\frac{1}{2}+...+\frac{1}{n}$$



Then by trapezoidal rule,

$$\int_1^n \frac{dx}{x} \approx \frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{2}\cdot\frac{1}{n} = S - \frac{1}{2} - \frac{1}{2n}$$

Since $\int_1^n \frac{dx}{x} = \ln n$, we have $S \approx \ln n + \frac{1}{2} + \frac{1}{2n}$. (Estimate is too low.)

3G-5 Referring to the two pictures above, one can see that if $f(x)$ is concave down on $[a, b]$, the trapezoidal rule gives too low an estimate; if $f(x)$ is concave up, the trapezoidal rule gives too high an estimate..

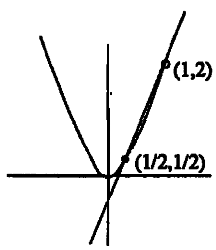
Unit 4. Applications of integration

4A. Areas between curves.

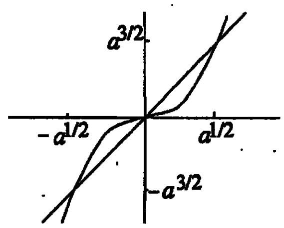
4A-1 a) $\int_{1/2}^{1}(3x-1-2x^2)dx = (3/2)x^2 - x - (2/3)x^3\Big|_{1/2}^{1} = 1/24$

b) $x^3 = ax \implies x = \pm a$ or $x = 0$. There are two enclosed pieces ($-a < x < 0$ and $0 < x < a$) with the same area by symmetry. Thus the total area is:

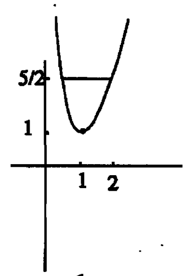
$$2\int_0^{\sqrt{a}}(ax - x^3)dx = ax^2 - (1/2)x^4\Big|_0^{\sqrt{a}} = a^2/2$$



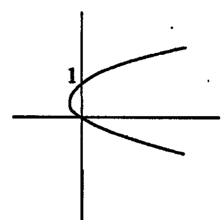
1a



1b



1c

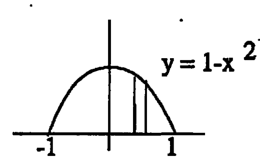


1d

c) $x + 1/x = 5/2 \implies x^2 + 1 = 5x/2 \implies x = 2$ or $1/2$. Therefore, the area is

$$\int_{1/2}^{2}[5/2 - (x + 1/x)]dx = 5x/2 - x^2/2 - \ln x\Big|_{1/2}^{2} = 15/8 - 2\ln 2$$

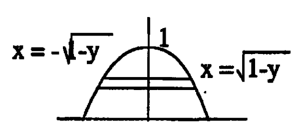
d) $\int_0^1 (y - y^2)dy = y^2/2 - y^3/3\Big|_0^1 = 1/6$



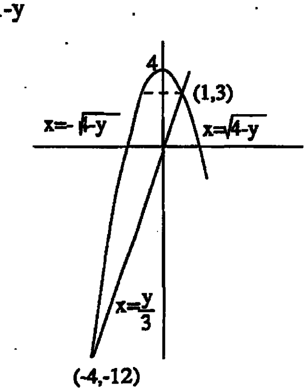
4A-2 First way (dx):

$$\int_{-1}^{1}(1-x^2)dx = 2\int_0^1(1-x^2)dx = 2x - 2x^3/3\Big|_0^1 = 4/3$$

Second way (dy): ($x = \pm\sqrt{1-y}$)



$$\int_0^1 2\sqrt{1-y}\,dy = (4/3)(1-y)^{3/2}\Big|_1^0 = 4/3$$



4A-3 $4 - x^2 = 3x \implies x = 1$ or -4. First way (dx):

$$\int_{-4}^{1}(4 - x^2 - 3x)dx = 4x - x^3/3 - 3x^2/2\Big|_{-4}^{1} = 125/6$$

Second way (dy): Lower section has area

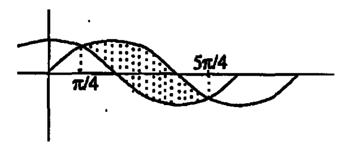
$$\int_{-12}^{3}(y/3 + \sqrt{4-y}\,dy = y^2/6 - (2/3)(4-y)^{3/2}\Big|_{-12}^{3} = 117/6$$

Upper section has area

$$\int_3^4 2\sqrt{4-y}\,dy = -(4/3)(4-y)^{3/2}\Big|_3^4 = 4/3$$

(See picture for limits of integration.) Note that $117/6 + 4/3 = 125/6$.

4A-4 $\sin x = \cos x \implies x = \pi/4 + k\pi$. So the area is



$$\int_{\pi/4}^{5\pi/4} (\sin x - \cos x)dx = (-\cos x - \sin x)\Big|_{\pi/4}^{5\pi/4} = 2\sqrt{2}$$

4B. Volumes by slicing; volumes of revolution

4B-1 a) $\displaystyle\int_{-1}^{1} \pi y^2 dx = \int_{-1}^{1} \pi(1-x^2)^2 dx = 2\pi\int_0^1 (1-2x^2+x^4)dx$

$= 2\pi(x - 2x^3/3 + x^5/5)\Big|_0^1 = 16\pi/15$

b) $\int_{-a}^{a} \pi y^2 dx = \int_{-a}^{a} \pi(a^2-x^2)^2 dx = 2\pi\int_0^a (a^4 - 2a^2x^2 + x^4)dx$

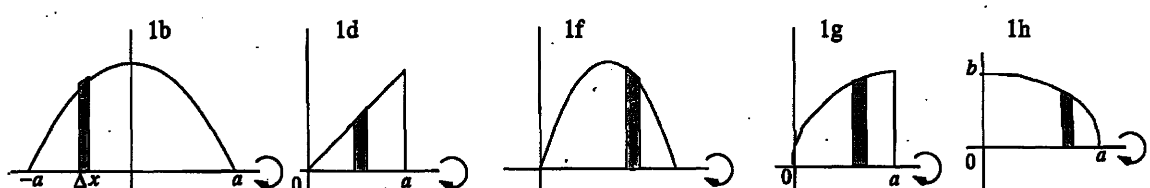
$= 2\pi(a^4x - 2a^2x^3/3 + x^5/5)\Big|_0^a = 16\pi a^5/15$

c) $\displaystyle\int_0^1 \pi x^2 dx = \pi/3$

d) $\displaystyle\int_0^a \pi x^2 dx = \pi a^3/3$

e) $\displaystyle\int_0^2 \pi(2x-x^2)^2 dx = \int_0^2 \pi(4x^2 - 4x^3 + x^4)dx = \pi(4x^3/3 - x^4 + x^5/5)\Big|_0^2 = 16\pi/15$

(Why (e) the same as (a)? Complete the square and translate.)



f) $\int_0^{2a} \pi(2ax-x^2)^2 dx = \int_0^{2a} \pi(4a^2x^2 - 4ax^3 + x^4)dx$

$= \pi(4a^2x^3/3 - ax^4 + x^5/5)\Big|_0^2 = 16\pi a^5/15$

(Why is (f) the same as (b)? Complete the square and translate.)

g) $\displaystyle\int_0^a axdx = \pi a^3/2$

h) $\displaystyle\int_0^a \pi y^2 dx = \int_0^a \pi b^2(1 - x^2/a^2)dx = \pi b^2(x - x^3/3a^2)\Big|_0^a = 2\pi b^2 a/3$

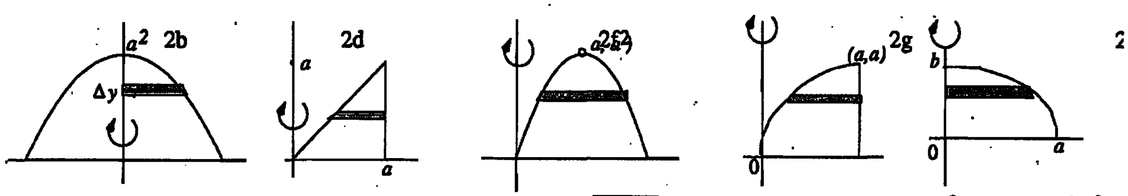
4B-2 a) $\displaystyle\int_0^1 \pi(1-y)dy = \pi/2$ b) $\displaystyle\int_0^{a^2} \pi(a^2-y)dy = \pi a^4/2$

c) $\displaystyle\int_0^1 \pi(1-y^2)dy = 2\pi/3$ d) $\displaystyle\int_0^a \pi(a^2-y^2)dy = 2\pi a^3/3$

e) $x^2 - 2x + y = 0 \implies x = 1 \pm \sqrt{1-y}$. Using the method of washers:

$$\int_0^1 \pi[(1+\sqrt{1-y})^2 - (1-\sqrt{1-y})^2]dy = \int_0^1 4\pi\sqrt{1-y}\,dy$$
$$= -(8/3)\pi(1-y)^{3/2}\Big|_0^1 = 8\pi/3$$

(In contrast with 1(e) and 1(a), rotation around the y-axis makes the solid in 2(e) different from 2(a).)



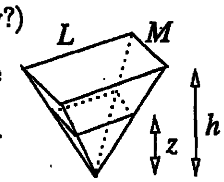
$x = \sqrt{a^2 - y}$ (for 2a, set a = 1) $x = y$ (for 2c, set a = 1) $x = a + \sqrt{a^2 - y}$ (for 2e, set a = 1) $x = y^2/a$ $x^2 = a^2(1 - y^2/b^2)$

f) $x^2 - 2ax + y = 0 \implies x = a \pm \sqrt{a^2 - y}$. Using the method of washers:

$$\int_0^{a^2} \pi[(a+\sqrt{a^2-y})^2 - (a-\sqrt{a^2-y})^2]dy = \int_0^{a^2} 4\pi a\sqrt{a^2-y}\,dy$$
$$= -(8/3)\pi a(a^2-y)^{3/2}\Big|_0^1 = 8\pi a^4/3$$

g) Using washers: $\displaystyle\int_0^a \pi(a^2 - (y^2/a)^2)dy = \pi(a^2y - y^5/5a^2)\Big|_0^a = 4\pi a^3/5.$

h) $\displaystyle\int_{-b}^b \pi x^2 dy = 2\pi\int_0^b a^2(1-y^2/b^2)dy = 2\pi(a^2y - a^2y^3/3b^2)\Big|_0^b = 4\pi a^2 b/3$ (The answer in 2(h) is double the answer in 1(h), with a and b reversed. Can you see why?)



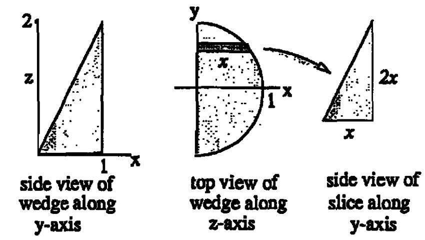
4B-3 Put the pyramid upside-down. By similar triangles, the base of the smaller bottom pyramid has sides of length $(z/h)L$ and $(z/h)M$.

The base of the big pyramid has area $b = LM$; the base of the smaller pyramid forms a cross-sectional slice, and has area

$$(z/h)L \cdot (z/h)M = (z/h)^2 LM = (z/h)^2 b$$

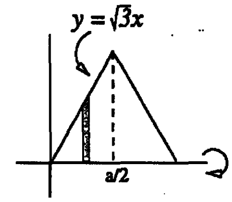
Therefore, the volume is

$$\int_0^h (z/h)^2 b\,dz = bz^3/3h^2\Big|_0^h = bh/3$$



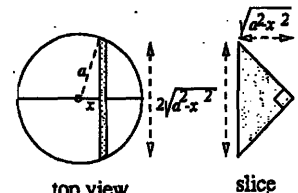
4B-4 The slice perpendicular to the xz-plane are right triangles with base of length x and height $z = 2x$. Therefore the area of a slice is x^2. The volume is

$$\int_{-1}^{1} x^2 dy = \int_{-1}^{1} (1-y^2)dy = 4/3$$



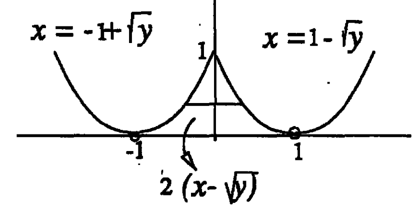
4B-5 One side can be described by $y = \sqrt{3}x$ for $0 \le x \le a/2$. Therefore, the volume is

$$2\int_0^{a/2} \pi y^2 dx 2 \int_0^{a/2} \pi(\sqrt{3}x)^2 dx = \pi a^3/4$$



4B-6 If the hypotenuse of an isoceles right triangle has length h, then its area is $h^2/4$. The endpoints of the slice in the xy-plane are $y = \pm\sqrt{a^2-x^2}$, so $h = 2\sqrt{a^2-x^2}$. In all the volume is

$$\int_{-a}^{a} (h^2/4)dx = \int_{-a}^{a} (a^2-x^2)dx = 4a^3/3$$



4B-7 Solving for x in $y = (x-1)^2$ and $y = (x+1)^2$ gives the values

$$x = 1 \pm \sqrt{y} \quad \text{and} \quad x = -1 \pm \sqrt{y}$$

The hard part is deciding which sign of the square root representing the endpoints of the square.

Method 1: The point $(0,1)$ has to be on the two curves. Plug in $y = 1$ and $x = 0$ to see that the square root must have the opposite sign from 1: $x = 1 - \sqrt{y}$ and $x = -1 + \sqrt{y}$.

Method 2: Look at the picture. $x = 1 + \sqrt{y}$ is the wrong choice because it is the right half of the parabola with vertex $(1,0)$. We want the left half: $x = 1 - \sqrt{y}$. Similarly, we want $x = -1 + \sqrt{y}$, the right half of the parabola with vertex $(-1,0)$. Hence, the side of the square is the interval $-1 + \sqrt{y} \le x \le 1 - \sqrt{y}$, whose length is $2(1-\sqrt{y})$, and the

$$\text{Volume} = \int_0^1 (2(1-\sqrt{y})^2 dy = 4\int_0^1 (1 - 2\sqrt{y} + y)dy = 2/3 .$$

4C. Volumes by shells

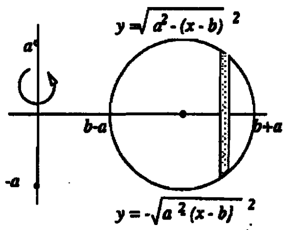
4C-1 a)

$$\text{Shells:} \quad \int_{b-a}^{b+a} (2\pi x)(2y)dx = \int_{b-a}^{b+a} 4\pi x\sqrt{a^2-(x-b)^2}dx$$

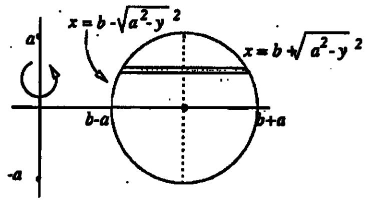
b) $(x-b)^2 = a^2 - y^2 \implies x = b \pm \sqrt{a^2-y^2}$

$$\text{Washers:} \quad \int_{-a}^{a} \pi(x_2^2 - x_1^2)dy = \int_{-a}^{a} \pi((b+\sqrt{a^2-y^2})^2 - (b-\sqrt{a^2-y^2})^2)dy$$

$$= \pi\int_{-a}^{a} 4b\sqrt{a^2-y^2}dy$$



Shells



Washers

c) $\displaystyle\int_{-a}^{a} \sqrt{a^2-y^2}\,dy = \pi a^2/2$, because it's the area of a semicircle of radius a.

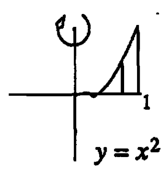
$$\text{Thus (b)} \implies \text{Volume of torus } = 2\pi^2 a^2 b$$

d) $z = x - b$, $dz = dx$

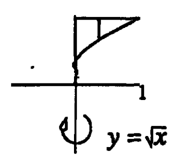
$$\int_{b-a}^{b+a} 4\pi x\sqrt{a^2-(x-b)^2}\,dx = \int_{-a}^{a} 4\pi(z+b)\sqrt{a^2-z^2}\,dz = \int_{-a}^{a} 4\pi b\sqrt{a^2-z^2}\,dz$$

because the part of the integrand with the factor z is odd, and so it integrates to 0.

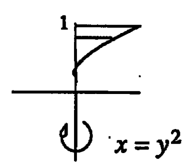
4C-2 $\displaystyle\int_0^1 2\pi xy\,dx = \int_0^1 2\pi x^3\,dx = \pi/2$



4C–2 (shells)



4C–3a (shells)



4C–3b (discs)

4C-3 Shells: $\displaystyle\int_0^1 2\pi x(1-y)\,dx = \int_0^1 2\pi x(1-\sqrt{x})\,dx = \pi/5$

Disks: $\displaystyle\int_0^1 \pi x^2\,dy = \int_0^1 \pi y^4\,dy = \pi/5$

4C-4 a) $\displaystyle\int_0^1 2\pi y(2x)\,dy = 4\pi\int_0^1 y\sqrt{1-y}\,dy$

b) $\displaystyle\int_0^{a^2} 2\pi y(2x)\,dy = 4\pi\int_0^{a^2} y\sqrt{a^2-y}\,dy$

c) $\displaystyle\int_0^1 2\pi y(1-y)\,dy$

d) $\displaystyle\int_0^a 2\pi y(a-y)\,dy$

e)
$$x^2-2x+y=0 \implies x=1\pm\sqrt{1-y}.$$

The interval $1-\sqrt{1-y}\le x\le 1+\sqrt{1-y}$ has length $2\sqrt{1-y}$.

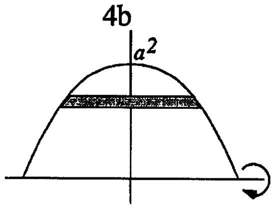
$$\implies V=\int_0^1 2\pi y(2\sqrt{1-y})dy = 4\pi\int_0^1 y\sqrt{1-y}dy$$

f)
$$x^2-2ax+y=0 \implies x=a\pm\sqrt{a^2-y}.$$

The interval $a-\sqrt{a^2-y}\le x\le a+\sqrt{a^2-y}$ has length $2\sqrt{a^2-y}$

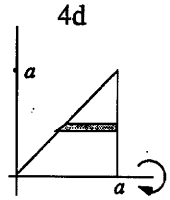
$$\implies V=\int_0^{a^2} 2\pi y(2\sqrt{a^2-y}dy = 4\pi\int_0^{a^2} y\sqrt{a^2-y}dy$$

4b



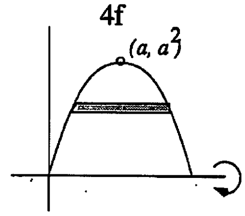
$x=\sqrt{a^2-y}$ *(right)*
$x=-\sqrt{a^2-y}$ *(left)*

4d



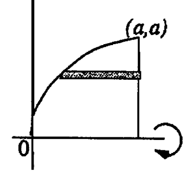
$x=y$

4f



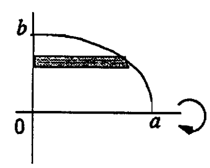
$x=a+\sqrt{a^2-y}$
$x=a-\sqrt{a^2-y}$

4g



$x=y^2/a$

4h



$x=a\sqrt{1-y^2/b^2}$

g) $\int_0^a 2\pi y(a-y^2/a)dy$

h) $\int_0^b 2\pi yxdy=\int_0^b 2\pi y(a^2(1-y^2/b^2)dy$

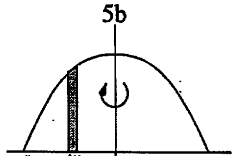
(Why is the lower limit of integration 0 rather than $-b$?)

4C-5 a) $\int_0^1 2\pi x(1-x^2)dx$ c) $\int_0^1 2\pi xydx=\int_0^1 2\pi x^2dx$

b) $\int_0^a 2\pi x(a^2-x^2)dx$ d) $\int_0^a 2\pi xydx=\int_0^a 2\pi x^2dx$

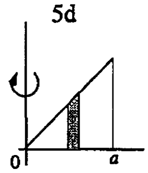
e) $\int_0^2 2\pi xydx=\int_0^2 2\pi x(2x-x^2)dx$

5b



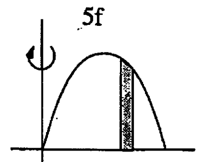
$y=a^2-x^2$
(for 5a, set $a=1$)

5d



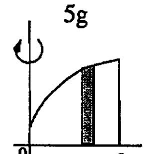
$y=x$
for 5c, set $a=1$)

5f



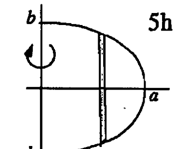
$y=2ax-x^2$
(for 5e, set $a=1$)

5g



$y=\sqrt{ax}$

5h

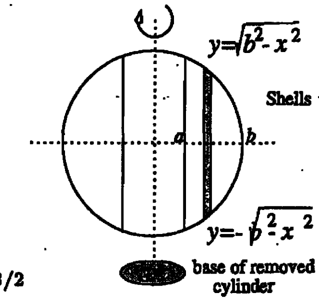


$y=b\sqrt{1-x^2/a^2}$
$y=-b\sqrt{1-x^2/a^2}$

f) $\int_0^{2a} 2\pi xy\,dx = \int_0^{2a} 2\pi x(ax - x^2)\,dx$ g) $\int_0^{a} 2\pi xy\,dx = \int_0^{a} 2\pi x\sqrt{ax}\,dx$

h) $\int_0^{a} 2\pi x(2y)\,dx = \int_0^{a} 2\pi x(2b^2(1 - x^2/a^2))\,dx$

(Why did y get doubled this time?)



4C-6

$$\int_a^b 2\pi x(2y)\,dx = \int_a^b 2\pi x(2\sqrt{b^2 - x^2})\,dx$$
$$= -(4/3)\pi(b^2 - x^2)^{3/2}\Big|_a^b = (4\pi/3)(b^2 - a^2)^{3/2}$$

4D. Average value

4D-1 Cross-sectional area at x is $= \pi y^2 = \pi \cdot (x^2)^2 = \pi x^4$. Therefore,

$$\text{average cross-sectional area } = \frac{1}{2}\int_0^2 \pi x^4\,dx = \frac{\pi x^5}{10}\Big|_0^2 = \frac{16\pi}{5}.$$

4D-2 Average $= \frac{1}{a}\int_a^{2a}\frac{dx}{x} = \frac{1}{a}\ln x\Big|_a^{2a} = \frac{1}{a}(\ln 2a - \ln a) = \frac{1}{a}\ln\left(\frac{2a}{a}\right) = \frac{\ln 2}{a}$.

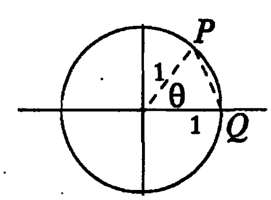
4D-3 Let $s(t)$ be the distance function; then the velocity is $v(t) = s'(t)$

$$\text{Average value of velocity} = \frac{1}{b-a}\int_a^b s'(t)\,dt = \frac{s(b) - s(a)}{b-a} \text{ by FT1}$$

= average velocity over time interval [a,b]

4D-4 By symmetry, we can restrict P to the upper semicircle.

By the law of cosines, we have $|PQ|^2 = 1^2 + 1^2 - 2\cos\theta$. Thus



$$\text{average of } |PQ|^2 = \frac{1}{\pi}\int_0^{\pi}(2 - 2\cos\theta)\,d\theta = \frac{1}{\pi}[2\theta - 2\sin\theta]_0^{\pi} = 2$$

(This is the value of $|PQ|^2$ when $\theta = \pi/2$, so the answer is reasonable.))

4D-5 By hypothesis, $g(x) = \frac{1}{x}\int_0^x f(t)\,dt$ To express $f(x)$ in terms of $g(x)$, multiply thourgh by x and apply the Sec. Fund. Thm:

$$\int_0^x f(t)\,dt = xg(x) \Rightarrow f(x) = g(x) + xg'(x) \text{ ,by FT2..}$$

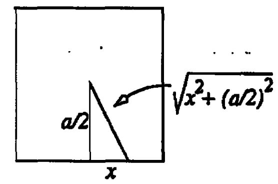
4D-6 Average value of $A(t) = \frac{1}{T}\int_0^T A_0 e^{rt}\,dt = \frac{1}{T}\frac{A_0}{r}e^{rt}|_0^T = \frac{A_0}{rT}(e^{rT} - 1)$

If rT is small, we can approximate: $e^{rT} \approx 1 + rT + \frac{(rT)^2}{2}$, so we get

$$A(t) \approx \frac{A_0}{rT}(rT + \frac{(rT)^2}{2}) = A_0(1 + \frac{rT}{2}).$$

(If $T \approx 0$, at the end of T years the interest added will be $A_0 rT$; thus the average is approximately what the account grows to in $T/2$ years, which seems reasonable.)

4D-7 $\frac{1}{b}\int_0^b x^2 dx = b^2/3$



4D-8 The average on each side is the same as the average over all four sides. Thus the average distance is

$$\frac{1}{a}\int_{-a/2}^{a/2} \sqrt{x^2 + (a/2)^2}dx$$

Can't be evaluated by a formula until Unit 5. The average of the square of the distance is

$$\frac{1}{a}\int_{-a/2}^{a/2} (x^2 + (a/2)^2)dx = \frac{2}{a}\int_0^{a/2} (x^2 + (a/2)^2)dx = a^2/3$$

4D-9 $\frac{1}{\pi/a}\int_0^{\pi/a} \sin ax\, dx \; -\frac{1}{\pi}\cos(ax)\Big|_0^{\pi/a} = 2/\pi$

4E. Parametric equations

4E-1 $y - x = t^2$, $y - 2x = -t$. Therefore,

$$y - x = (y - 2x)^2 \implies y^2 - 4xy + 4x^2 - y + x = 0 \quad \text{(parabola)}$$

4E-2 $x^2 = t^2 + 2 + 1/t^2$ and $y^2 = t^2 - 2 + 1/t^2$. Subtract, getting the hyperbola $x^2 - y^2 = 4$

4E-3 $(x - 1)^2 + (y - 4)^2 = \sin^2\theta + \cos^2 t = 1$ (circle)

4E-4 $1 + \tan^2 t = \sec^2 t \implies 1 + x^2 = y^2$ (hyperbola)

4E-5 $x = \sin 2t = 2\sin t\cos t = \pm 2\sqrt{1 - y^2}y$. This gives $x^2 = 4y^2 - 4y^4$.

4E-6 $y' = 2x$, so $t = 2x$ and

$$x = t/2, \quad y = t^2/4$$

4E-7 Implicit differentiation gives $2x + 2yy' = 0$, so that $y' = -x/y$. So the parameter is $t = -x/y$. Substitute $x = -ty$ in $x^2 + y^2 = a^2$ to get

$$t^2y^2 + y^2 = a^2 \implies y^2 = a^2/(1 + t^2)$$

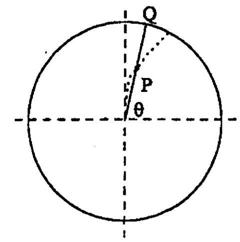
Thus

$$y = \frac{a}{\sqrt{1+t^2}}, \quad x = \frac{-at}{\sqrt{1+t^2}}$$

For $-\infty < t < \infty$, this parametrization traverses the upper semicircle $y > 0$ (going clockwise). One can also get the lower semicircle (also clockwise) by taking the negative square root when solving for y,

$$y = \frac{-a}{\sqrt{1+t^2}}, \quad x = \frac{at}{\sqrt{1+t^2}}$$

4E-8 The tip Q of the hour hand is given in terms of the angle θ by $Q = (\cos\theta, \sin\theta)$ (units are meters).



Next we express θ in terms of the time parameter t (hours). We have

$$\theta = \left\{ \begin{matrix} \pi/2, t=0 \\ \pi/3, t=1 \end{matrix} \right\} \theta \text{ decreases linearly with t}$$

$$\Longrightarrow \theta - \frac{\pi}{2} = \frac{\frac{\pi}{3} - \frac{\pi}{2} \cdot (t-0)}{1-0}$$. Thus we get $\theta = \frac{\pi}{2} - \frac{\pi}{6}t$.

Finally, for the snail's position P, we have

$P = (t\cos\theta, t\sin\theta)$, where t increases from 0 to 1. So,

$x = t\cos(\frac{\pi}{2} - \frac{\pi}{6}t) = t\sin\frac{\pi}{6}t, \qquad y = t\sin(\frac{\pi}{2} - \frac{\pi}{6}t) = t\cos\frac{\pi}{6}t$

4F. Arclength

4F-1 a) $ds = \sqrt{1+(y')^2}dx = \sqrt{26}dx$. Arclength $= \int_0^1 \sqrt{26}dx = \sqrt{26}$.

b) $ds = \sqrt{1+(y')^2}dx = \sqrt{1+(9/4)x}dx$.

$$\text{Arclength} = \int_0^1 \sqrt{1+(9/4)x}dx = (8/27)(1+9x/4)^{3/2}\Big|_0^1 = (8/27)((13/4)^{3/2}-1)$$

c) $y' = -x^{-1/3}(1-x^{2/3})^{1/2} = -\sqrt{x^{-2/3}-1}$. Therefore, $ds = x^{-1/3}dx$, and

$$\text{Arclength} = \int_0^1 x^{-1/3}dx = (3/2)x^{2/3}\Big|_0^1 = 3/2$$

d) $y' = x(2+x^2)^{1/2}$. Therefore, $ds = \sqrt{1+2x^2+x^4}dx = (1+x^2)dx$ and

$$\text{Arclength} = \int_1^2 (1+x^2)dx = x + x^3/3\Big|_1^2 = 10/3$$

4F-2 $y' = (e^x - e^{-x})/2$, so the hint says $1+(y')^2 = y^2$ and $ds = \sqrt{1+(y')^2}dx = ydx$. Thus,

$$\text{Arclength} = (1/2)\int_0^b (e^x + e^{-x})dx = (1/2)(e^x - e^{-x})\Big|_0^b = (e^b - e^{-b})/2$$

4F-3 $y' = 2x$, $\sqrt{1+(y')^2} = \sqrt{1+4x^2}$. Hence, arclength $= \int_0^b \sqrt{1+4x^2}dx$. **4F-4** $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}dt = \sqrt{4t^2+9t^4}dt$. Therefore,

$$\begin{aligned} \text{Arclength} &= \int_0^2 \sqrt{4t^2+9t^4}dt = \int_0^2 (4+9t^2)^{1/2}tdt \\ &= (1/27)(4+9t^2)^{3/2}\Big|_0^2 = (40^{3/2}-8)/27 \end{aligned}$$

4F-5 $dx/dt = 1 - 1/t^2$, $dy/dt = 1 + 1/t^2$. Thus

$$ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}dt = \sqrt{2 + 2/t^4}dt \quad \text{and}$$

$$\text{Arclength} = \int_1^2 \sqrt{2 + 2/t^4}dt$$

4F-6 a) $dx/dt = 1 - \cos t$, $dy/dt = \sin t$.

$$ds/dt = \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{2 - 2\cos t} \quad \text{(speed of the point)}$$

Forward motion (dx/dt) is largest for t an odd multiple of π ($\cos t = -1$). Forward motion is smallest for t an even multiple of π ($\cos t = 1$). *(continued →)*

Remark: The largest forward motion is when the point is at the top of the wheel and the smallest is when the point is at the bottom (since $y = 1 - \cos t$.)

b) $$\int_0^{2\pi} \sqrt{2 - 2\cos t}dt = \int_0^{2\pi} 2\sin(t/2)dt = -4\cos(t/2)|_0^{2\pi} = 8$$

4F-7 $\displaystyle\int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}dt$

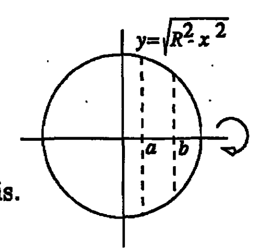
4F-8 $dx/dt = e^t(\cos t - \sin t)$, $dy/dt = e^t(\cos t + \sin t)$.

$$ds = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2}dt = e^t\sqrt{2\cos^2 t + 2\sin^2 t}dt = \sqrt{2}e^t dt$$

Therefore, the arclength is

$$\int_0^{10} \sqrt{2}e^t dt = \sqrt{2}(e^{10} - 1)$$

4G. Surface Area



4G-1 The curve $y = \sqrt{R^2 - x^2}$ for $a \le x \le b$ is revolved around the x-axis.

Since we have $y' = -x/\sqrt{R^2 - x^2}$, we get

$$ds = \sqrt{1 + (y')^2}dx = \sqrt{1 + x^2/(R^2 - x^2)}dx = \sqrt{R^2/(R^2 - x^2)}dx = (R/y)dx$$

Therefore, the area element is

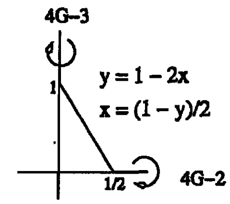
$$dA = 2\pi y ds = 2\pi R dx$$

and the area is

$$\int_a^b 2\pi R dx = 2\pi R(b - a)$$

4G-2 Limits are $0 \le x \le 1/2$. $ds = \sqrt{5}dx$, so

$$dA = 2\pi y ds = 2\pi(1-2x)\sqrt{5}dx \implies A = 2\pi\sqrt{5}\int_0^{1/2} (1-2x)dx = \sqrt{5}\pi/2$$



4G-3 Limits are $0 \le y \le 1$. $x = (1-y)/2$; $dx/dy = -1/2$. Thus

$$ds = \sqrt{1 + (dx/dy)^2}dy = \sqrt{5/4}dy;$$

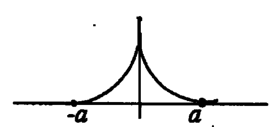
$$dA = 2\pi y ds = \pi(1-y)(\sqrt{5}/2)dx \implies A = (\sqrt{5}\pi/2)\int_0^1 (1-y)dy = \sqrt{5}\pi/4$$

4G-4 $A = \displaystyle\int 2\pi y ds = \int_0^4 2\pi x^2\sqrt{1+4x^2}dx$

4G-5 $x = \sqrt{y}$, $dx/dy = -1/2\sqrt{y}$; and $ds = \sqrt{1 + 1/4y}dy$

$$\begin{aligned} A &= \int 2\pi x ds = \int_0^2 2\pi\sqrt{y}\sqrt{1+1/4y}dy \\ &= \int_0^2 2\pi\sqrt{y + 1/4}dy \\ &= (4\pi/3)(y+1/4)^{3/2}\Big|_0^2 = (4\pi/3)((9/4)^{3/2} - (1/4)^{3/2}) \\ &= 13\pi/3 \end{aligned}$$

4G-6 $y = (a^{2/3} - x^{2/3})^{3/2} \implies y' = -x^{-1/3}(a^{2/3} - x^{2/3})^{1/2}$. Hence

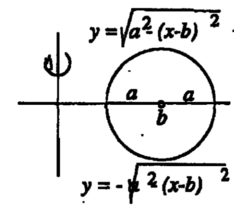


$y = (a^{2/3} - x^{2/3})^{3/2}$

$$ds = \sqrt{1 + x^{-2/3}(a^{2/3} - x^{2/3})}dx = a^{1/3}x^{-1/3}dx$$

Therefore, (using symmetry on the interval $-a \le x \le a$)

$$\begin{aligned} A &= \int 2\pi y ds = 2\int_0^a 2\pi(a^{2/3} - x^{2/3})^{3/2}a^{1/3}x^{-1/3}dx \\ &= (4\pi)(2/5)(-3/2)a^{1/3}(a^{2/3} - x^{2/3})^{5/2}\Big|_0^a \\ &= (12\pi/5)a^2 \end{aligned}$$



upper and lower surfaces are symmetrical and equal

4G-7 a) Top half: $y = \sqrt{a^2 - (x-b)^2}$, $y' = (b-x)/y$. Hence,

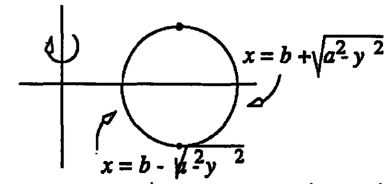
$$ds = \sqrt{1 + (b-x)^2/y^2}dx = \sqrt{(y^2 + (b-x)^2)/y^2}dx = (a/y)dx$$

Since we are only covering the top half we double the integral for area:

$$A = \int 2\pi x ds = 4\pi a\int_{b-a}^{b+a}\frac{xdx}{\sqrt{a^2-(x-b)^2}}$$

b) We need to rotate two curves $x_2 = b + \sqrt{a^2 - y^2}$ and $x_1 = b - \sqrt{a^2 - y^2}$ around the y-axis. The value

$$dx_2/dy = -(dx_1/dy) = -y/\sqrt{a^2 - y^2}$$



inner and outer surfaces are not symmetrical and not equal

So in both cases,

$$ds = \sqrt{1 + y^2/(a^2 - y^2)}dy = (a/\sqrt{a^2 - y^2})dy$$

The integral is

$$A = \int 2\pi x_2 ds + \int 2\pi x_1 ds = \int_{-a}^{a} 2\pi(x_1 + x_2)\frac{a dy}{\sqrt{a^2 - y^2}}$$

But $x_1 + x_2 = 2b$, so

$$A = 4\pi ab \int_{-a}^{a} \frac{dy}{\sqrt{a^2 - y^2}}$$

c) Substitute $y = a\sin\theta$, $dy = a\cos\theta d\theta$ to get

$$A = 4\pi ab \int_{-\pi/2}^{\pi/2} \frac{a\cos\theta d\theta}{a\cos\theta} = 4\pi ab \int_{-\pi/2}^{\pi/2} d\theta = 4\pi^2 ab$$

4H. Polar coordinate graphs

4H-1 We give the polar coordinates in the form (r, θ):

a) $(3, \pi/2)$ b) $(2, \pi)$ c) $(2, \pi/3)$ d) $(2\sqrt{2}, 3\pi/4)$
e) $(\sqrt{2}, -\pi/4$ or $7\pi/4)$ f) $(2, -\pi/2$ or $3\pi/2)$
g) $(2, -\pi/6$ or $11\pi/6)$ h) $(2\sqrt{2}, -3\pi/4$ or $5\pi/4)$

4H-2 a) (i) $(x-a)^2 + y^2 = a^2 \Rightarrow x^2 - 2ax + y^2 = 0 \Rightarrow r^2 - 2ar\cos\theta = 0 \Rightarrow r = 2a\cos\theta$.

(ii) $\angle OPQ = 90^\circ$, since it is an angle inscribed in a semicircle.
In the right triangle OPQ, $|OP| = |OQ|\cos\theta$, i.e., $r = 2a\cos\theta$.

b) (i) Analogous to 4H-2a(i); ans: $r = 2a\sin\theta$.
(ii) analogous to 4H-2a(ii); note that $\angle OQP = \theta$, since both angles are complements of $\angle POQ$.

c) (i) OQP is a right triangle, $|OP| = r$, and $\angle POQ = \alpha - \theta$.
The polar equation is $r\cos(\alpha - \theta) = a$, or in expanded form,

$$r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = a, \quad \text{or finally,}$$

$$\frac{x}{A} + \frac{y}{B} = 1,$$

since from the right triangles OAQ and OBQ, we have $\cos\alpha = \dfrac{a}{A}$, $\sin\alpha = \cos BOQ = \dfrac{a}{B}$.

d) Since $|OQ| = \sin\theta$, we have:
if P is above the x-axis, $\sin\theta > 0$, $OP| = |OQ| - |QR|$, or $r = a - a\sin\theta$;
if P is below the x-axis, $\sin\theta < 0$, $OP| = |OQ| + |QR|$, or $r = a + a|\sin\theta| = a - a\sin\theta$.
Thus the equation is $r = a(1 - \sin\theta)$.

e) Briefly, when $P = (0,0)$, $|PQ||PR| = a \cdot a = a^2$, the constant.
Using the law of cosines,

$$|PR|^2 = r^2 + a^2 - 2ar\cos\theta;$$
$$|PQ|^2 = r^2 + a^2 - 2ar\cos(\pi - \theta) = r^2 + a^2 + 2ar\cos\theta$$

Therefore

$$|PQ|^2|PR|^2 = (r^2 + a^2)^2 - (2ar\cos\theta)^2 = (a^2)^2$$

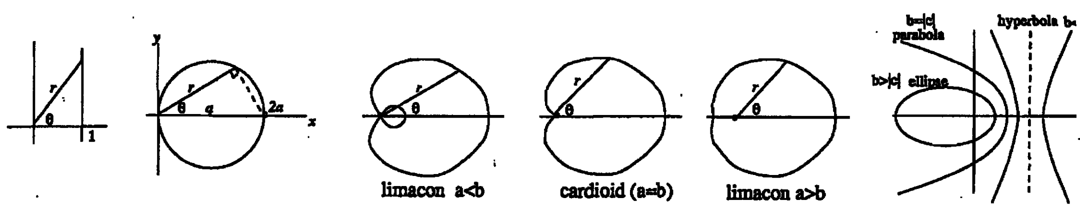
which simplifies to

$$r^2 = 2a^2\cos 2\theta.$$

4H-3 a) $r = \sec\theta \Longrightarrow r\cos\theta = 1 \Longrightarrow x = 1$ b) $r = 2a\cos\theta \Longrightarrow r^2 = r \cdot 2a\cos\theta = 2ax \Longrightarrow x^2 + y^2 = 2ax$

c) $r = (a + b\cos\theta)$ (This figure is a cardiod for $a = b$, a limaçon with a loop for $0 < a < b$, and a limaçon without a loop for $a > b > 0$.)

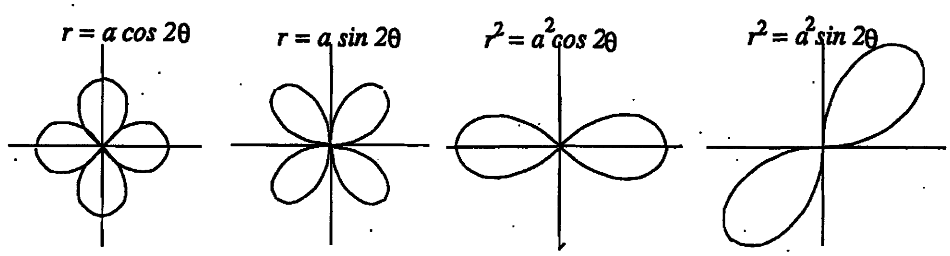
$$r^2 = ar + br \cdot \cos\theta = ar + bx \Longrightarrow x^2 + y^2 = a\sqrt{x^2 + y^2} + bx$$



8a 8b 8c 8d

(d)
$$\begin{aligned} r = a/(b + c\cos\theta) &\Longrightarrow r(b + c\cos\theta) = a \Longrightarrow rb + cx = a \\ &\Longrightarrow rb = a - cx \Longrightarrow r^2b^2 = a^2 - 2acx + c^2x^2 \\ &\Longrightarrow a^2 - 2acx + (c^2 - b^2)x^2 - b^2y^2 = 0 \end{aligned}$$

(e)
$$\begin{aligned} r = a\sin(2\theta) &\Longrightarrow r = 2a\sin\theta\cos\theta = 2axy/r^2 \\ &\Longrightarrow r^3 = 2axy \Longrightarrow (x^2 + y^2)^{3/2} = 2axy \end{aligned}$$



f) $r = a\cos(2\theta) = a(2\cos^2\theta - 1) = a(\dfrac{2x^2}{x^2 + y^2} - 1) \Longrightarrow (x^2 + y^2)^{3/2} = a(x^2 - y^2)$

g) $r^2 = a^2\sin(2\theta) = 2a^2\sin\theta\cos\theta = 2a^2\dfrac{xy}{r^2} \Longrightarrow r^4 = 2a^2xy \Longrightarrow (x^2 + y^2)^2 = 2axy$

h) $r^2 = a^2\cos(2\theta) = a^2(\dfrac{2x^2}{x^2 + y^2} - 1) \Longrightarrow (x^2 + y^2)^2 = a^2(x^2 - y^2)$

i) $r = e^{a\theta} \Longrightarrow \ln r = a\theta \Longrightarrow \ln\sqrt{x^2 + y^2} = a\tan^{-1}\dfrac{y}{x}$

4I. Area and arclength in polar coordinates

4I-1 $\sqrt{(dr/d\theta)^2 + r^2}\,d\theta$

a) $\sec^2\theta\, d\theta$

b) $2a\,d\theta$

c) $\sqrt{a^2 + b^2 + 2ab\cos\theta}\,d\theta$

d) $\dfrac{a\sqrt{b^2 + c^2 + 2bc\cos\theta}}{(b + c\cos\theta)^2}d\theta$

e) $a\sqrt{4\cos^2(2\theta) + \sin^2(2\theta)}\,d\theta$

f) $a\sqrt{4\sin^2(2\theta) + \cos^2(2\theta)}\,d\theta$

g) Use implicit differentiation:

$$2rr' = 2a^2\cos(2\theta) \implies r' = a^2\cos(2\theta)/r \implies (r')^2 = a^2\cos^2(2\theta)/\sin(2\theta)$$

Hence, using a common denominator and $\cos^2 + \sin^2 = 1$,

$$ds = \sqrt{a^2\cos^2(2\theta)/\sin(2\theta) + a^2\sin(2\theta)}\,d\theta = \frac{a}{\sqrt{\sin(2\theta)}}d\theta$$

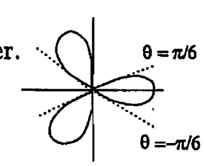
h) This is similar to (g):

$$ds = \frac{a}{\sqrt{\cos(2\theta)}}d\theta$$

i) $\sqrt{1 + a^2}e^{a\theta}d\theta$

4I-2 $dA = (r^2/2)d\theta$. The main difficulty is to decide on the endpoints of integration. Endpoints are successive times when $r = 0$.

$$\cos(3\theta) = 0 \implies 3\theta = \pi/2 + k\pi \implies \theta = \pi/6 + k\pi/3, \quad k \text{ an integer.}$$

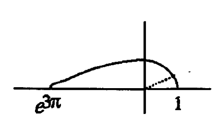


three-leaf rose
three empty sectors

$$\text{Thus,} \quad A = \int_{-\pi/6}^{\pi/6} (a^2\cos^2(3\theta)/2)d\theta = a^2\int_0^{\pi/6}\cos^2(3\theta)d\theta.$$

(Stop here in Unit 4. Evaluated in Unit 5.)

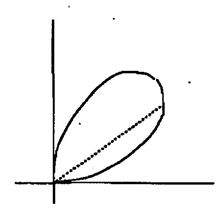
4I-3 $A = \int (r^2/2)d\theta = \int_0^{\pi}(e^{6\theta}/2)d\theta = (1/12)e^{6\theta}\big|_0^{\pi} = (e^{6\pi} - 1)/12$

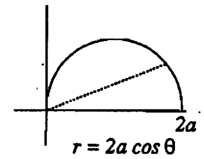


4I-4 Endpoints are successive time when $r = 0$.

$$\sin(2\theta) = 0 \implies 2\theta = k\pi, \quad k \text{ an integer.}$$

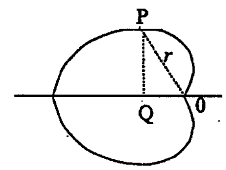
Thus, $A = \int (r^2/2)d\theta = \int_0^{\pi/2}(a^2/2)\sin(2\theta)d\theta = -(a^2/4)\cos(2\theta)\big|_0^{\pi/2} = a^2/2.$





4I-5 $r = 2a\cos\theta$, $ds = 2ad\theta$, $-\pi/2 < \theta < \pi/2$. (The range was chosen carefully so that $r > 0$.) Total length of the circle is $2\pi a$. Since the upper and lower semicircles are symmetric, it suffices to calculate the average over the upper semicircle:

$$\frac{1}{\pi a}\int_0^{\pi/2} 2a\cos\theta(2a)d\theta = \frac{4a}{\pi}\sin\theta\Big|_0^{\pi/2} = \frac{4a}{\pi}$$



4I-6 a) Since the upper and lower halves of the cardiod are symmetric, it suffices to calculate the average distance to the x-axis just for a point on the upper half. We have $r = a(1-\cos\theta)$, and the distance to the x-axis is $r\sin\theta$, so

$$\frac{1}{\pi}\int_0^{\pi} r\sin\theta d\theta = \frac{1}{\pi}\int_0^{\pi} a(1-\cos\theta)\sin\theta d\theta = \frac{a}{2\pi}(1-\cos\theta)^2\Big|_0^{\pi} = \frac{2a}{\pi}$$

(b)
$$ds = \sqrt{(dr/d\theta)^2 + r^2}d\theta = a\sqrt{(1-\cos\theta)^2 + \sin^2\theta}d\theta$$
$$= a\sqrt{2 - 2\cos\theta}d\theta = 2a\sin(\theta/2)d\theta, \qquad \text{using the half angle formula.}$$

$$\text{arclength} = \int_0^{2\pi} 2a\sin(2\theta)d\theta = -4a\cos(\theta/2)|_0^{2\pi} = 8a$$

For the average, don't use the half-angle version of the formula for ds, and use the interval $-\pi < \theta < \pi$, where $\sin\theta$ is odd:

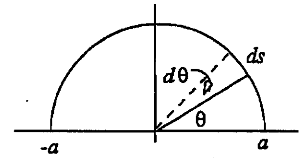
$$\text{Average} = \frac{1}{8a}\int_{-\pi}^{\pi} |r\sin\theta| a\sqrt{2-2\cos\theta}d\theta = \frac{1}{8a}\int_{-\pi}^{\pi} |\sin\theta|\sqrt{2}a^2(1-\cos\theta)^{3/2}d\theta$$
$$= \frac{\sqrt{2}a}{4}\int_0^{\pi}(1-\cos\theta)^{3/2}\sin\theta d\theta = \frac{\sqrt{2}a}{10}(1-\cos\theta)^{5/2}\Big|_0^{\pi} = \frac{4}{5}a$$

4I-7 $dx = -a\sin\theta d\theta$. So the semicircle $y > 0$ has area

$$\int_{-a}^{a} y dx = \int_{\pi}^{0} a\sin\theta(-a\sin\theta)d\theta = a^2\int_0^{\pi}\sin^2\theta d\theta$$

But
$$\int_0^{\pi}\sin^2\theta d\theta = \frac{1}{2}\int_0^{\pi}(1-\cos(2\theta)d\theta = \pi/2$$

So the area is $\pi a^2/2$ as it should be for a semicircle.



Arclength: $ds^2 = dx^2 + dy^2$

$\Longrightarrow (ds)^2 = (-a\sin\theta d\theta)^2 + (a\cos\theta d\theta)^2 = a^2(\sin^2 d\theta + \cos^2 d\theta)(d\theta)^2$

$\Longrightarrow ds = ad\theta$ (obvious from picture).

$$\int ds = \int_0^{2\pi} ad\theta = 2\pi a$$

4J. Other applications

4J-1 Divide the water in the hole into n equal circular discs of thickness Δy.

Volume of each disc: $\pi\left(\frac{1}{2}\right)^2 \Delta y$.

Energy to raise the disc of water at depth y_i to surface: $\frac{\pi}{4} k y_i \Delta y$.

Adding up the energies for the different discs, and passing to the limit,

$$E = \lim_{n\to\infty} \sum_1^n \frac{\pi}{4} k y_i \Delta y = \int_0^{100} \frac{\pi}{4} ky\, dy = \frac{\pi k}{4} \frac{y^2}{2}\Big]_0^{100} = \frac{\pi k 10^4}{8}.$$

4J-2 Divide the hour into n equal small time intervals Δt.

At time t_i, $i = 1, \ldots, n$, there are $x_0 e^{-kt_i}$ grams of material, producing approximately $r x_0 e^{-kt_i} \Delta t$ radiation units over the time interval $[t_i, t_i + \Delta t]$.

Adding and passing to the limit,

$$R = \lim_{n\to\infty} \sum_1^n r\, x_0 e^{-kt_i} \Delta t = \int_0^{60} r\, x_0 e^{-kt}\, dt = r\, x_0 \frac{e^{-kt}}{-k}\Big]_0^{60} = \frac{r\, x_0}{k}\left(1 - e^{-60k}\right).$$

4J-3 Divide up the pool into n thin concentric cylindrical shells, of radius r_i, $i = 1, \ldots, n$, and thickness Δr.

The volume of the i-th shell is approximately $2\pi\, r_i D\, \Delta r$.

The amount of chemical in the i-th shell is approximately $\frac{k}{1+r_i^2} 2\pi\, r_i D\, \Delta r$.

Adding, and passing to the limit,

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_1^n \frac{k}{1+r_i^2} 2\pi\, r_i D\, \Delta r = \int_0^R 2\pi k D \frac{r}{1+r^2}\, dr \\ &= \pi k D \ln(1+r^2)\Big]_0^R = \pi k D \ln(1+R^2) \text{ gms.} \end{aligned}$$

4J-4 Divide the time interval into n equal small intervals of length Δt by the points t_i, $i = 1, \ldots, n$.

The approximate number of heating units required to maintain the temperature at 75° over the time interval $[t_i, t_i + \Delta t]$: is

$$\left[75 - 10\left(6 - \cos\frac{\pi t_i}{12}\right)\right] \cdot k\, \Delta t.$$

Adding over the time intervals and passing to the limit:

$$\begin{aligned} \text{total heat} &= \lim_{n\to\infty} \sum_1^n \left[75 - 10\left(6 - \cos\frac{\pi t_i}{12}\right)\right] \cdot k\, \Delta t \\ &= \int_0^{24} k\left[75 - 10\left(6 - \cos\frac{\pi t}{12}\right)\right] dt \\ &= \int_0^{24} k\left(15 + 10\cos\frac{\pi t}{12}\right) dt = k\left[15t + \frac{120}{\pi}\sin\frac{\pi t}{12}\right]_0^{24} = 360k. \end{aligned}$$

4J-5 Divide the month into n equal intervals of length Δt by the points t_i, $i = 1, \dots, n$. Over the time interval $[t_i.t_i + \Delta t]$, the number of units produced is about $(10 + t_i)\,\Delta t$. The cost of holding these in inventory until the end of the month is $c(30 - t_i)(10 + t_i)\,\Delta t$. Adding and passing to the limit,

$$\begin{aligned} \text{total cost} &= \lim_{n\to\infty} \sum_{1}^{n} c(30 - t_i)(10 + t_i)\,\Delta t \\ &= \int_0^{30} c(30 - t)(10 + t)\,dt \;=\; c\left[300t + 10t^2 - \frac{t^3}{3}\right]_0^{30} \;=\; 9000c. \end{aligned}$$

Unit 5. Integration techniques

5A. Inverse trigonometric functions; Hyperbolic functions

5A-1 a) $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$ b) $\sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$

c) $\tan\theta = 5$ implies $\sin\theta = 5/\sqrt{26}$, $\cos\theta = 1/\sqrt{26}$, $\cot\theta = 1/5$, $\csc\theta = \sqrt{26}/5$, $\sec\theta = \sqrt{26}$ (from triangle)

d) $\sin^{-1}\cos(\frac{\pi}{6}) = \sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$ e) $\tan^{-1}\tan(\frac{\pi}{3}) = \frac{\pi}{3}$

f) $\tan^{-1}\tan(\frac{2\pi}{3}) = \tan^{-1}\tan(\frac{-\pi}{3}) = \frac{-\pi}{3}$ g) $\lim\limits_{x\to-\infty}\tan^{-1}x = \frac{-\pi}{2}$.

5A-2 a) $\displaystyle\int_1^2 \frac{dx}{x^2+1} = \tan^{-1}x\Big|_1^2 = \tan^{-1}2 - \frac{\pi}{4}$

b) $\displaystyle\int_b^{2b} \frac{dx}{x^2+b^2} = \int_b^{2b}\frac{d(by)}{(by)^2+b^2}$ (put $x = by$) $\displaystyle = \int_1^2 \frac{dy}{b(y^2+1)} = \frac{1}{b}(\tan^{-1}2 - \frac{\pi}{4})$

c) $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x\Big|_{-1}^1 = \frac{\pi}{2} - \frac{-\pi}{2} = \pi$

5A-3 a) $y = \dfrac{x-1}{x+1}$, so $1-y^2 = 4x/(x+1)^2$, and $\dfrac{1}{\sqrt{1-y^2}} = \dfrac{(x+1)}{2\sqrt{x}}$. Hence

$$\begin{aligned}\frac{dy}{dx} &= \frac{2}{(x+1)^2}\\ \frac{d}{dx}\sin^{-1}y &= \frac{dy/dx}{\sqrt{1-y^2}}\\ &= \frac{2}{(x+1)^2}\cdot\frac{(x+1)}{2\sqrt{x}}\\ &= \frac{1}{(x+1)\sqrt{x}}\end{aligned}$$

b) $\operatorname{sech}^2 x = 1/\cosh^2 x = 4/(e^x + e^{-x})^2$

c) $y = x + \sqrt{x^2+1}$, $dy/dx = 1 + x/\sqrt{x^2+1}$.

$$\frac{d}{dx}\ln y = \frac{dy/dx}{y} = \frac{1 + x/\sqrt{x^2+1}}{x+\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}$$

d) $\cos y = x \implies (-\sin y)(dy/dx) = 1$

$$\frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-x^2}}$$

e) Chain rule:

$$\frac{d}{dx}\sin^{-1}(x/a) = \frac{1}{\sqrt{1-(x/a)^2}}\cdot\frac{1}{a} = \frac{1}{\sqrt{a^2-x^2}}$$

f) Chain rule:

$$\frac{d}{dx}\sin^{-1}(a/x) = \frac{1}{\sqrt{1-(a/x)^2}}\cdot\frac{-a}{x^2} = \frac{-a}{x\sqrt{x^2-a^2}}$$

g) $y = x/\sqrt{1-x^2}$, $dy/dx = (1-x^2)^{-3/2}$, $1+y^2 = 1/(1-x^2)$. Thus

$$\frac{d}{dx}\tan^{-1}y = \frac{dy/dx}{1+y^2} = (1-x^2)^{-3/2}(1-x^2) = \frac{1}{\sqrt{1-x^2}}$$

Why is this the same as the derivative of $\sin^{-1}x$?

h) $y = \sqrt{x-1}$, $dy/dx = -1/2\sqrt{x-1}$, $1-y^2 = x$. Thus,

$$\frac{d}{dx}\sin^{-1}y = \frac{dy/dx}{\sqrt{1-y^2}} = \frac{-1}{2\sqrt{x(1-x)}}$$

5A-4 a) $y' = \sinh x$. A tangent line through the origin has the equation $y = mx$. If it meets the graph at $x = a$, then $ma = \cosh(a)$ and $m = \sinh(a)$. Therefore, $a\sinh(a) = \cosh(a)$.

b) Take the difference:

$$F(a) = a\sinh(a) - \cosh(a)$$

Newton's method for finding $F(a) = 0$, is the iteration

$$a_{n+1} = a_n - F(a_n)/F'(a_n) = a_n - \tanh(a_n) + 1/a_n$$

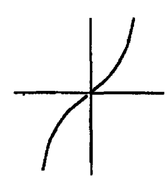
With $a_1 = 1$, $a_2 = 1.2384$, $a_3 = 1.2009$, $a_4 = 1.19968$. A serviceable approximation is

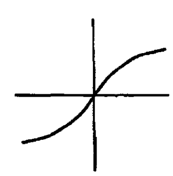
$$a \approx 1.2$$

(The slope is $m = \sinh(a) \approx 1.5$.) The functions F and y are even. By symmetry, there is another solution $-a$ with slope $-\sinh a$.

5A-5 a)

$$y = \sinh x = \frac{e^x - e^{-x}}{2}$$
$$y' = \cosh x = \frac{e^x + e^{-x}}{2}$$
$$y'' = \sinh x$$


$y = sinh\, x$


$y = sinh^{-1}x$

y' is never zero, so no critical points. Inflection point $x = 0$; slope of y is 1 there. y is an odd function, like $e^x/2$ for $x >> 0$.

b) $y = \sinh^{-1}x \iff x = \sinh y$. Domain is the whole x-axis.

c) Differentiate $x = \sinh y$ implicitly with respect to x:

$$1 = \cosh y \cdot \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{\sinh^2 y + 1}}$$
$$\frac{d\sinh^{-1}x}{dx} = \frac{1}{\sqrt{x^2+1}}$$

d)

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{dx}{a\sqrt{x^2+a^2/a^2}}$$
$$= \int \frac{d(x/a)}{\sqrt{(x/a)^2+1}}$$
$$= \sinh^{-1}(x/a) + c$$

5A-6 a) $\frac{1}{\pi}\int_0^{\pi} \sin\theta d\theta = 2/\pi$

b) $y = \sqrt{1-x^2} \implies y' = -x/\sqrt{1-x^2} \implies 1+(y')^2 = 1/(1-x^2)$. Thus

$$ds = w(x)dx = dx/\sqrt{1-x^2}.$$

Therefore the average is

$$\int_{-1}^{1} \sqrt{1-x^2}\frac{dx}{\sqrt{1-x^2}} \Big/ \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$$

The numerator is $\int_{-1}^{1} dx = 2$. To see that these integrals are the same as the ones in part (a), take $x = \cos\theta$ (as in polar coordinates). Then $dx = -\sin\theta d\theta$ and the limits of integral are from $\theta = \pi$ to $\theta = 0$. Reversing the limits changes the minus back to plus:

$$\int_{-1}^{1} \sqrt{1-x^2}\frac{dx}{\sqrt{1-x^2}} = \int_0^{\pi} \sin\theta d\theta$$
$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \int_0^{\pi} d\theta = \pi$$

(The substitution $x = \sin t$ works similarly, but the limits of integration are $-\pi/2$ and $\pi/2$.)

c) ($x = \sin t$, $dx = \cos t dt$)

$$\frac{1}{2}\int_{-1}^{1} \sqrt{1-x^2}dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos^2 t dt = \int_0^{\pi/2} \cos^2 t dt$$
$$= \int_0^{\pi/2} \frac{1+\cos 2t}{2} dt$$
$$= \pi/4$$

5B. Integration by direct substitution

Do these by guessing and correcting the factor out front. The substitution used implicitly is given alongside the answer.

5B-1 $\int x\sqrt{x^2-1}dx = \frac{1}{3}(x^2-1)^{\frac{3}{2}} + c$ ($u = x^2-1$, $du = 2xdx$)

5B-2 $\int e^{8x}dx = \frac{1}{8}e^{8x} + c$ $(u = 8x,\ du = 8dx)$

5B-3 $\int \frac{\ln x dx}{x} = \frac{1}{2}(\ln x)^2 + c$ $(u = \ln x,\ du = dx/x)$

5B-4 $\int \frac{\cos x dx}{2 + 3\sin x} = \frac{\ln(2 + 3\sin x)}{3} + c$ $(u = 2 + 3\sin x,\ du = 3\cos x dx)$

5B-5 $\int \sin^2 x \cos x dx = \frac{\sin x^3}{3} + c$ $(u = \sin x,\ du = \cos x dx)$

5B-6 $\int \sin 7x dx = \frac{-\cos 7x}{7} + c$ $(u = 7x,\ du = 7dx)$

5B-7 $\int \frac{6x dx}{\sqrt{x^2+4}} = 6\sqrt{x^2+4} + c$ $(u = x^2 + 4,\ du = 2x dx)$

5B-8 Use $u = \cos(4x)$, $du = -4\sin(4x)dx$,

$$\int \tan 4x dx = \int \frac{\sin(4x)dx}{\cos(4x)} = \int \frac{-du}{4u}$$
$$= -\frac{\ln u}{4} + c = -\frac{\ln(\cos 4x)}{4} + c$$

5B-9 $\int e^x(1+e^x)^{-1/3}dx = \frac{3}{2}(1+e^x)^{2/3} + c$ $(u = 1 + e^x,\ du = e^x dx)$

5B-10 $\int \sec 9x dx = \frac{1}{9}\ln(\sec(9x) + \tan(9x)) + c$ $(u = 9x,\ du = 9dx)$

5B-11 $\int \sec^2 9x dx = \frac{\tan 9x}{9} + c$ $(u = 9x,\ du = 9dx)$

5B-12 $\int xe^{-x^2}dx = \frac{-e^{-x^2}}{2} + c$ $(u = x^2,\ du = 2x dx)$

5B-13 $u = x^3$, $du = 3x^2 dx$ implies

$$\int \frac{x^2 dx}{1+x^6} = \int \frac{du}{3(1+u^2)} = \frac{\tan^{-1} u}{3} + c$$
$$= \frac{\tan^{-1}(x^3)}{3} + c$$

5B-14 $\int_0^{\pi/3} \sin^3 x \cos x dx = \int_{\sin 0}^{\sin \pi/3} u^3 du$ $(u = \sin x,\ du = \cos x dx)$

$$= \int_0^{\sqrt{3}/2} u^3 du = u^4/4\Big|_0^{\sqrt{3}/2} = \tfrac{9}{64}$$

5B-15 $\int_1^e \frac{(\ln x)^{3/2}dx}{x} = \int_{\ln 1}^{\ln e} u^{3/2}du$ $(u = \ln x,\ du = dx/x)$

$$= \int_0^1 y^{3/2} dy = (2/5) y^{5/2} \Big|_0^1 = \frac{2}{5}$$

5B-16 $\displaystyle\int_{-1}^{1} \frac{\tan^{-1} x dx}{1+x^2} = \int_{\tan^{-1}(-1)}^{\tan^{-1} 1} u du$ $(u = \tan^{-1} x,\ du = dx/(1+x^2))$

$$= \int_{-\pi/4}^{\pi/4} u du = \frac{u^2}{2}\Big|_{-\pi/4}^{\pi/4} = 0$$

($\tan x$ is odd and hence $\tan^{-1} x$ is also odd, so the integral had better be 0)

5C. Trigonometric integrals

5C-1 $\displaystyle\int \sin^2 x dx = \int \frac{1-\cos 2x}{2} dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$

5C-2 $\displaystyle\int \sin^3(x/2) dx = \int (1-\cos^2(x/2)) \sin(x/2) dx = \int -2(1-u^2) du$
(put $u = \cos(x/2)$, $du = (-1/2)\sin(x/2) dx$)

$$= -2u + \frac{2u^3}{3} + c = -2\cos(x/2) + \frac{2\cos(x/2)^3}{3} + c$$

5C-3 $\displaystyle\int \sin^4 x dx = \int (\frac{1-\cos 2x}{2})^2 dx = \int \frac{1 - 2\cos 2x + \cos^2 2x}{4} dx$

$$\int \frac{\cos^2(2x)}{4} dx = \int \frac{1+\cos 4x}{8} dx = \frac{x}{8} + \frac{\sin 4x}{32} + c$$

Adding together all terms:

$$\int \sin^4 x dx = \frac{3x}{8} - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c$$

5C-4 $\displaystyle\int \cos^3(3x) dx = \int (1-\sin^2(3x)) \cos(3x) dx = \int \frac{1-u^2}{3} du$ $(u = \sin(3x),\ du = 3\cos(3x) dx)$

$$= \frac{u}{3} - \frac{u^3}{9} + c = \frac{\sin(3x)}{3} - \frac{\sin(3x)^3}{9} + c$$

5C-5 $\displaystyle\int \sin^3 x \cos^2 x dx = \int (1-\cos^2 x)\cos^2 x \sin x dx = \int -(1-u^2) u^2 dy$ $(u = \cos x,\ du = -\sin x dx)$

$$= -\frac{u^3}{3} + \frac{u^5}{5} + c = -\frac{\cos x^3}{3} + \frac{\cos x^5}{5} + c$$

5C-6 $\displaystyle\int \sec^4 x dx = \int (1+\tan^2 x)\sec^2 x dx = \int (1+u^2) du$ $(u = \tan x,\ du = \sec^2 x dx)$

$$= u + \frac{u^3}{3} + c = \tan x + \frac{\tan^3 x}{3} + c$$

5C-7 $\displaystyle\int \sin^2(4x)\cos^2(4x) dx = \int \frac{\sin^2 8x dx}{4} = \int \frac{(1-\cos 16x) dx}{8} = \frac{1}{8} - \frac{\sin 16x}{128} + c$

A slower way is to use

$$\sin^2(4x)\cos^2(4x) = \left(\frac{1-\cos(8x)}{2}\right)\left(\frac{1+\cos(8x)}{2}\right)$$

multiply out and use a similar trick to handle $\cos^2(8x)$.

5C-8

$$\begin{aligned}\int \tan^2(ax)\cos(ax)dx &= \int \frac{\sin^2(ax)}{\cos(ax)}dx \\ &= \int \frac{1-\cos^2(ax)}{\cos(ax)}dx \\ &= \int (\sec(ax)-\cos(ax))dx \\ &= \frac{1}{a}\ln(\sec(ax)+\tan(ax)) - \frac{1}{a}\sin(ax) + c\end{aligned}$$

5C-9

$$\begin{aligned}\int \sin^3 x \sec^2 x dx &= \int \frac{1-\cos^2 x}{\cos^2 x}\sin x dx \\ &= \int -\frac{1-u^2}{u^2}du \qquad (u=\cos x,\ du = -\sin x dx) \\ &= u + \frac{1}{u} + c = \cos x + \sec x + c\end{aligned}$$

5C-10

$$\begin{aligned}\int (\tan x + \cot x)^2 dx = \int \tan^2 x + 2 + \cot^2 x dx &= \int \sec^2 x + \csc^2 x dx \\ = \tan x - \cot x + c\end{aligned}$$

5C-11 $\int \sin x \cos(2x)dx$

$$= \int \sin x(2\cos^2 x - 1)dx = \int (1-2u^2)du \ (u=\cos x,\ du - \sin x dx)$$

$$= u - \frac{2}{3}u^3 + c = \cos x - \frac{2}{3}\cos^3 x + c$$

5C-12 $\displaystyle\int_0^\pi \sin x\cos(2x)dx = \cos x - \frac{2}{3}\cos^3 x\Big|_0^\pi = \frac{-2}{3}$ (See 27.)

5C-13 $ds = \sqrt{1+(y')^2}dx = \sqrt{1+\cot^2 x}dx = \csc x dx.$

$$\text{arclength } = \int_{\pi/4}^{\pi/2} \csc x dx = -\ln(\csc x + \cot x)\Big|_{\pi/4}^{\pi/2} = \ln(1+\sqrt{2})$$

5C-14 $\displaystyle\int_0^{\pi/a} \pi\sin^2(ax)dx = \pi\int_0^{\pi/a}(1/2)(1-\cos(2ax))dx = \pi^2/2a$

5D. Integration by inverse substitution

5D-1 Put $x = a\sin\theta$, $dx = a\cos\theta d\theta$:

$$\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{1}{a^2}\int \sec^2\theta d\theta = \frac{1}{a^2}\tan\theta + c = \frac{x}{a^2\sqrt{a^2-x^2}} + c$$

5D-2 Put $x = a\sin\theta$, $dx = a\cos\theta d\theta$:

$$\int \frac{x^3dx}{\sqrt{a^2-x^2}} = a^3\int \sin^3\theta d\theta = a^3\int (1-\cos^2\theta)\sin\theta d\theta$$
$$= a^3(-\cos\theta + (1/3)\cos^3\theta) + c$$
$$= -a^2\sqrt{a^2-x^2} + (a^2-x^2)^{3/2}/3 + c$$

5D-3 By direct substitution $(u = 4+x^2)$,

$$\int \frac{xdx}{4+x^2} = (1/2)\ln(4+x^2) + c$$

Put $x = 2\tan\theta$, $dx = 2\sec^2\theta d\theta$;

$$\int \frac{dx}{4+x^2} = \frac{1}{2}\int d\theta = \theta/2 + c$$

In all,

$$\int \frac{(x+1)dx}{4+x^2} = (1/2)\ln(4+x^2) + (1/2)\tan^{-1}(x/2) + c$$

5D-4 Put $x = a\sinh y$, $dx = a\cosh y dy$. Since $1+\sinh^2 y = \cosh^2 y$,

$$\int \sqrt{a^2+x^2}dx = a^2\int \cosh^2 y dy = \frac{a^2}{2}\int (\cosh(2y)-1)dy$$
$$= (a^2/4)\sinh(2y) - a^2y/2 + c = (a^2/2)\sinh y\cosh y - a^2y/2 + c$$
$$= x\sqrt{a^2+x^2}/2 - a^2\sinh^{-1}(x/a) + c$$

5D-5 Put $x = a\sin\theta$, $dx = a\cos\theta d\theta$:

$$\int \frac{\sqrt{a^2-x^2}dx}{x^2} = \int \cot^2\theta d\theta$$
$$= \int (\csc^2\theta - 1)d\theta = -\ln(\csc\theta + \cot\theta) - \theta + c$$
$$= -\ln(a/x + \sqrt{a^2-x^2}/x) - \sin^{-1}(x/a) + c$$

5D-6 Put $x = a\sinh y$, $dx = a\cosh y dy$.

$$\int x^2\sqrt{a^2+x^2}dx = a^4\int \sinh^2 y\cosh^2 y dy$$
$$= (a^4/2)\int \sinh^2(2y)dy = a^4/4\int (\cosh(4y)-1)dy$$
$$= (a^4/16)\sinh(4y) - a^4y/4 + c$$
$$= (a^4/8)\sinh(2y)\cosh(2y) - a^4y/4 + c$$
$$= (a^4/4)\sinh y\cosh y(\cosh^2 y + \sinh^2 y) - a^4y/4 + c$$
$$= (1/4)x\sqrt{a^2+x^2}(2x^2+a^2) - (a^4/4)\sinh^{-1}(x/a) + c$$

5D-7 Put $x = a\sec\theta$, $dx = a\sec\theta\tan\theta d\theta$:

$$\begin{aligned}\int\frac{\sqrt{x^2-a^2}dx}{x^2} &= \int\frac{\tan^2\theta d\theta}{\sec\theta}\\ &= \int\frac{(\sec^2\theta-1)d\theta}{\sec\theta} = \int(\sec\theta-\cos\theta)d\theta\\ &= \ln(\sec\theta+\tan\theta)-\sin\theta+c\\ &= \ln(x/a+\sqrt{x^2-a^2}/a)-\sqrt{x^2-a^2}/x+c\\ &= \ln(x+\sqrt{x^2-a^2})-\sqrt{x^2-a^2}/x+c_1 \quad (c_1 = c-\ln a)\end{aligned}$$

5D-8 Short way: $u = x^2-9$, $du = 2xdx$,

$$\int x\sqrt{x^2-9}dx = (1/3)(x^2-9)^{3/2}+c \quad \text{direct substitution}$$

Long way (method of this section): Put $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta d\theta$.

$$\begin{aligned}\int x\sqrt{x^2-9}dx &= 27\int\sec^2\theta\tan^2\theta d\theta\\ &= 27\int\tan^2\theta d(\tan\theta) = 9\tan^3\theta+c\\ &= (1/3)(x^2-9)^{3/2}+c\end{aligned}$$

($\tan\theta = \sqrt{x^2-9}/3$). The trig substitution method does not lead to a dead end, but it's not always fastest.

5D-9 $y' = 1/x$, $ds = \sqrt{1+1/x^2}dx$, so

$$\text{arclength } = \int_1^b\sqrt{1+1/x^2}dx$$

Put $x = \tan\theta$, $dx = \sec^2\theta d\theta$,

$$\begin{aligned}\int\frac{\sqrt{x^2+1}dx}{x} &= \int\frac{\sec\theta}{\tan\theta}\sec^2\theta d\theta\\ &= \int\frac{\sec\theta(1+\tan^2\theta)}{\tan\theta}d\theta\\ &= \int(\csc\theta+\sec\theta\tan\theta)d\theta\\ &= -\ln(\csc\theta+\cot\theta)+\sec\theta+c\\ &= -\ln(\sqrt{x^2+1}/x+1/x)+\sqrt{x^2+1}+c\\ &= -\ln(\sqrt{x^2+1}+1)+\ln x+\sqrt{x^2+1}+c\end{aligned}$$

$$\text{arclength } = -\ln(\sqrt{b^2+1}+1)+\ln b+\sqrt{b^2+1}+\ln(\sqrt{2}+1)-\sqrt{2}$$

Completing the square

5D-10 $\displaystyle\int \frac{dx}{(x^2+4x+13)^{3/2}} = \int \frac{dx}{((x+2)^2+3^2)^{3/2}}$ $(x+2 = 3\tan\theta,\ dx = 3\sec^2\theta d\theta)$

$$= \frac{1}{9}\int \cos\theta d\theta = \frac{1}{9}\sin\theta + c = \frac{(x+2)}{9\sqrt{x^2+4x+13}} + c$$

5D-11

$$\int x\sqrt{-8+6x-x^2}dx = \int x\sqrt{1-(x-3)^2}dx \quad (x-3=\sin\theta,\ dx=\cos\theta d\theta)$$
$$= \int (\sin\theta+3)\cos^2\theta d\theta$$
$$= (-1/3)\cos^3\theta + (3/2)\int(\cos 2\theta+1)d\theta$$
$$= -(1/3)\cos^3\theta + (3/4)\sin 2\theta + (3/2)\theta + c$$
$$= -(1/3)\cos^3\theta + (3/2)\sin\theta\cos\theta + (3/2)\theta + c$$
$$= -(1/3)(-8+6x-x^2)^{3/2}$$
$$+ (3/2)(x-3)\sqrt{-8+6x-x^2} + (3/2)\sin^{-1}(x-3) + c$$

5D-12

$$\int \sqrt{-8+6x-x^2}dx = \int \sqrt{1-(x-3)^2}dx \quad (x-3=\sin\theta,\ dx=\cos\theta d\theta)$$
$$= \int \cos^2\theta d\theta$$
$$= \frac{1}{2}\int(\cos 2\theta+1)d\theta$$
$$= \frac{1}{4}\sin 2\theta + \frac{\theta}{2} + c$$
$$= \frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2} + c$$
$$= \frac{(x-3)\sqrt{-8+6x-x^2}}{2} + \frac{\sin^{-1}(x-3)}{2} + c$$

5D-13 $\displaystyle\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$. Put $x-1=\sin\theta$, $dx=\cos\theta d\theta$.

$$= \int d\theta = \theta + c = \sin^{-1}(x-1) + c$$

5D-14 $\displaystyle\int \frac{xdx}{\sqrt{x^2+4x+13}} = \int \frac{xdx}{\sqrt{(x+2)^2+3^2}}$. Put $x+2=3\tan\theta$, $dx=3\sec^2\theta$.

$$= \int (3\tan\theta-2)\sec\theta d\theta = 3\sec\theta - 2\ln(\sec\theta+\tan\theta) + c$$
$$= \sqrt{x^2+4x+13} - 2\ln(\sqrt{x^2+4x+13}/3 + (x+2)/3) + c$$
$$= \sqrt{x^2+4x+13} - 2\ln(\sqrt{x^2+4x+13} + (x+2)) + c_1 \quad (c_1 = c - \ln 3)$$

5D-15 $\displaystyle\int \frac{\sqrt{4x^2-4x+17}dx}{2x-1} = \int \frac{\sqrt{(2x-1)^2+4^2}dx}{2x-1}$
(put $2x-1=4\tan\theta$, $dx = 2\sec^2\theta d\theta$ as in Problem 9)

$$
\begin{aligned}
&= 2\int \frac{\sec\theta}{\tan\theta}\sec^2\theta d\theta \\
&= 2\int \frac{\sec\theta(1+\tan^2\theta)}{\tan\theta} d\theta \\
&= 2\int (\csc\theta + \sec\theta\tan\theta) d\theta \\
&= -2\ln(\csc\theta+\cot\theta) + 2\sec\theta + c \\
&= -2\ln(\sqrt{4x^2-4x+17}/(2x-1) + 4/(2x-1)) + \sqrt{4x^2-4x+17}/2 + c \\
&= -2\ln(\sqrt{4x^2-4x+17}+4) + 2\ln(2x-1) + \sqrt{4x^2-4x+17}/2 + c
\end{aligned}
$$

5E. Integration by partial fractions

5E-1 $\displaystyle\frac{1}{(x-2)(x+3)} = \frac{1/5}{x-2} + \frac{-1/5}{x+3}$ (cover up)

$$\int \frac{dx}{(x-2)(x+3)} = (1/5)\ln(x-2) - (1/5)\ln(x+3) + c$$

5E-2 $\displaystyle\frac{x}{(x-2)(x+3)} = \frac{2/5}{x-2} + \frac{3/5}{x+3}$ (cover up)

$$\int \frac{xdx}{(x-2)(x+3)} = (2/5)\ln(x-2) + (3/5)\ln(x+3) + c$$

5E-3 $\displaystyle\frac{x}{(x-2)(x+2)(x+3)} = \frac{1/10}{x-2} + \frac{1/2}{x+2} + \frac{-3/5}{x+3}$ (cover up)

$$\int \frac{xdx}{(x^2-4)(x+3)} = (1/10)\ln(x-2) + (1/2)\ln(x+2) - (3/5)\ln(x+3)$$

5E-4 $\displaystyle\frac{3x^2+4x-11}{(x^2-1)(x-2)}dx = \frac{2}{x-1} + \frac{-2}{x+1} + \frac{3}{x-2}$ (cover-up)

$$\int \frac{2dx}{x-1} + \frac{-2dx}{x+1} + \frac{3dx}{x-2} = 2\ln(x-1) - 2\ln(x+1) + 3\ln(x-2) + c$$

5E-5 $\dfrac{3x+2}{x(x+1)^2} = \dfrac{2}{x} + \dfrac{B}{x+1} + \dfrac{1}{(x+1)^2}$ (coverup); to get B, put say $x = 1$:

$$\frac{5}{4} = 2 + \frac{B}{2} + \frac{1}{4} \implies B = -2$$

$$\int \frac{3x+2}{x(x+1)^2}dx = 2\ln x - 2\ln(x+1) - \frac{1}{x+1} + c$$

5E-6 $\dfrac{2x-9}{(x^2+9)(x+2)} = \dfrac{Ax+B}{x^2+9} + \dfrac{C}{x+2}$
By cover-up, $C = -1$. To get B and A,

$$x = 0 \implies \frac{-9}{9\cdot 2} = \frac{B}{9} - \frac{1}{2} \implies B = 0$$

$$x = 1 \implies \frac{-7}{10\cdot 3} = \frac{A}{10} - \frac{1}{3} \implies A = 1$$

$$\int \frac{2x-9}{(x^2+9)(x+2)}dx = \frac{1}{2}\ln(x^2+9) - \ln(x+2) + c$$

5E-7 Instead of thinking of (4) as arising from (1) by multiplication by $x-1$, think of it as arising from

$$x - 7 = A(x+2) + B(x-1)$$

by division by $x+2$; since this new equation is valid for all x, the line (4) will be valid for $x \neq -2$, in particular it will be valid for $x = 1$.

5E-8 Long division:

a) $\dfrac{x^2}{x^2-1} = 1 + \dfrac{1}{x^2-1}$

b) $\dfrac{x^3}{x^2-1} = x + \dfrac{x}{x^2-1}$

c) $\dfrac{x^2}{3x-1} = x/3 + 1/9 + \dfrac{1/9}{3x-1}$

d) $\dfrac{x+2}{3x-1} = \dfrac{1}{3} + \dfrac{7/3}{3x-1}$

e) $\dfrac{x^8}{(x+2)^2(x-2)^2} = A_4x^4 + A_3x^3 + A_2x^2 + A_1x + A_0 + \dfrac{B_3x^3 + B_2x^2 + B_1x + B_0}{(x+2)^2(x-2)^2}$

5E-9 a) Cover-up gives

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{1/2}{x-1} + \frac{-1/2}{x+1}$$

From 8a,

$$\frac{x^2}{x^2-1} = 1 + \frac{1/2}{x-1} + \frac{-1/2}{x+1} \quad \text{and}$$

$$\int \frac{x^2dx}{x^2-1} = x + (1/2)\ln(x-1) - (1/2)\ln(x+1) + c$$

b) Cover-up gives

$$\frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)} = \frac{1/2}{x-1} + \frac{1/2}{x+1}$$

From 8b,

$$\frac{x^3}{x^2-1} = x + \frac{1/2}{x-1} + \frac{1/2}{x+1} \quad \text{and}$$

$$\int \frac{x^3 dx}{x^2-1} = x^2/2 + (1/2)\ln(x-1) + (1/2)\ln(x+1) + c$$

c) From 8c,

$$\int \frac{x^2}{3x-1} dx = x^2/6 + x/9 + (1/27)\ln(3x-1) + c$$

d) From 8d,

$$\int \frac{x+2}{3x-1} dx = x/3 + (7/9)\ln(3x-1)$$

e) Cover-up says that the proper rational function will be written as

$$\frac{a_1}{x-2} + \frac{a_2}{(x-2)^2} + \frac{b_1}{x+2} + \frac{b_2}{(x+2)^2}$$

where the coefficients a_2 and b_2 can be evaluted from the B's using cover-up and the coefficients a_1 and b_1 can then be evaluated using $x = 0$ and $x = 1$, say. Therefore, the integral has the form

$$A_4x^5/5 + A_3x^4/4 + A_2x^3/3 + A_1x^2/2 + A_0x + c$$
$$+ a_1\ln(x-2) - \frac{a_2}{x-2} + b_1\ln(x+2) - \frac{b_2}{x+2}$$

5E-10 a) By cover-up,

$$\frac{1}{x^3-x} = \frac{1}{x(x-1)(x+1)} = \frac{-1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1}$$

$$\int \frac{dx}{x^3-x} = -\ln x + \frac{1}{2}\ln(x-1) + \frac{1}{2}\ln(x+1) + c$$

b) By cover-up, $\dfrac{(x+1)}{(x-2)(x-3)} = \dfrac{-3}{x-2} + \dfrac{4}{x-3}$. Therefore,

$$\int \frac{(x+1)}{(x-2)(x-3)} dx = -3\ln(x-2) + 4\ln(x-3) + c$$

c) $\dfrac{(x^2+x+1)}{x^2+8x} = 1 + \dfrac{-7x+1}{x^2+8x}$. By cover-up,

$$\frac{-7x+1}{x^2+8x} = \frac{-7x+1}{x(x+8)} = \frac{1/8}{x} + \frac{-57/8}{x+8} \quad \text{and}$$

$$\int \frac{(x^2+x+1)}{x^2+8x} = x + (1/8)\ln x - (57/8)\ln(x+8) + c$$

d) Seeing double? It must be late.

e) $\dfrac{1}{x^3+x^2} = \dfrac{1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

Use the cover-up method to get $B = 1$ and $C = 1$. For A,

$$x = 1 \implies \frac{1}{2} = A + 1 + \frac{1}{2} \implies A = -1$$

In all,

$$\int \frac{dx}{x^3+x^2} = \int \left(-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right) dx = -\ln x + \ln(x+1) - \frac{1}{x} + c$$

f) $\dfrac{x^2+1}{x^3+2x^2+x} = \dfrac{x^2+1}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

By cover-up, $A = 1$ and $C = -2$. For B,

$$x = 1 \implies \frac{2}{4} = 1 + \frac{B}{2} - \frac{2}{4} \implies B = 0 \quad \text{and}$$

$$\int \frac{x^2+1}{x^3+2x^2+x} dx = \int \left(\frac{1}{x} - \frac{2}{(x+1)^2}\right) dx = \ln x + \frac{2}{x+1} + c$$

g) Multiply out denominator: $(x+1)^2(x-1) = x^3+x^2-x-1$. Divide into numerator:

$$\frac{x^3}{x^3+x^2-x-1} = 1 + \frac{-x^2+x+1}{x^3+x^2-x-1}$$

Write the proper rational function as

$$\frac{-x^2+x+1}{(x+1)^2(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$

By cover-up, $B = 1/2$ and $C = 1/4$. For A,

$$x = 0 \implies -1 = A + \frac{1}{2} - \frac{1}{4} \implies A = -\frac{5}{4} \quad \text{and}$$

$$\begin{aligned}\int \frac{x^3}{(x+1)^2(x-1)} dx &= \int \left(1 + \frac{-5/4}{x+1} + \frac{1/2}{(x+1)^2} + \frac{1/4}{x-1}\right) dx \\ &= x - (5/4)\ln(x+1) - \frac{1}{2(x+1)} + (1/4)\ln(x-1) + c\end{aligned}$$

h) $\displaystyle\int \frac{(x^2+1)dx}{x^2+2x+2} = \int (1 - \frac{1+2x}{x^2+2x+2}) dx = x - \int \frac{(2y-1)dy}{y^2+1}$ (put $y = x+1$)

$$\begin{aligned}&= x - \ln(y^2+1) + \tan^{-1} y + c \\ &= x - \ln(x^2+2x+2) + \tan^{-1}(x+1) + c\end{aligned}$$

5E-11 Separate:

$$\frac{dy}{y(1-y)} = dx$$

Expand using partial fractions and integrate

$$\int \left(\frac{1}{y} - \frac{1}{y-1}\right) dy = \int dx$$

Hence,

$$\ln y - \ln(y-1) = x + c$$

Exponentiate:

$$\frac{y}{y-1} = e^{x+c} = Ae^x \quad (A = e^c)$$

$$y = \frac{Ae^x}{Ae^x - 1}$$

(If you integrated $1/(1-y)$ to get $-\ln(1-y)$ then you arrive at

$$y = \frac{Ae^x}{Ae^x + 1}$$

This is the same family of answers with A and $-A$ traded.)

5E-12 a) $1 + z^2 = 1 + \tan^2(\theta/2) = \sec^2(\theta/2)$. Therefore,

$$\cos^2(\theta/2) = \frac{1}{1+z^2} \quad \text{and} \quad \sin^2(\theta/2) = 1 - \frac{1}{1+z^2} = \frac{z^2}{1+z^2}$$

Next,

$$\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2) = \frac{1}{1+z^2} - \frac{z^2}{1+z^2} = \frac{1-z^2}{1+z^2} \quad \text{and}$$

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2) = 2\sqrt{\frac{1}{1+z^2}}\sqrt{\frac{z^2}{1+z^2}} = \frac{2z}{1+z^2}$$

Finally,

$$dz = (1/2)\sec^2(\theta/2)d\theta = (1/2)(1+z^2)d\theta \implies d\theta = \frac{2dz}{1+z^2}$$

b)

$$\begin{aligned}
\int_0^\pi \frac{d\theta}{1+\sin\theta} &= \int_{\tan 0}^{\tan \pi/2} \frac{2dz/(1+z^2)}{1+2z/(1+z^2)} \\
&= \int_0^\infty \frac{2dz}{z^2+1+2z} = \int_0^\infty \frac{2dz}{(z+1)^2} \\
&= \left.\frac{-2}{1+z}\right|_0^\infty = 2
\end{aligned}$$

c)

$$\int_0^{\pi} \frac{d\theta}{(1+\sin\theta)^2} = \int_{\tan 0}^{\tan \pi/2} \frac{2dz/(1+z^2)}{(1+2z/(1+z^2))^2} = \int_0^{\infty} \frac{2(1+z^2)dz}{(1+z)^4}$$
$$= \int_1^{\infty} \frac{2(1+(y-1)^2)dy}{y^4} \quad (\text{put } y = z+1)$$
$$= \int_1^{\infty} \frac{(2y^2-4y+4)dy}{y^4} = \int_1^{\infty} (2y^{-2}-4y^{-3}+4y^{-4})dy$$
$$= -2y^{-1}+2y^{-2}-(4/3)y^{-3}\Big|_1^{\infty} = 4/3$$

d)

$$\int_0^{\pi} \sin\theta d\theta = \int_0^{\infty} \frac{2z}{1+z^2}\frac{2dz}{1+z^2} = \int_0^{\infty} \frac{4zdz}{(1+z^2)^2}$$
$$= \frac{-2}{1+z^2}\Big|_0^{\infty} = 2$$

5E-13 a) $z = \tan(\theta/2) \implies 1+\cos\theta = 2/(1+z^2)$ and $0 \le \theta \le \pi/2$ corresponds to $0 \le z \le 1$.

$$A = \int_0^{\pi/2} \frac{d\theta}{2(1+\cos\theta)^2} = \int_0^1 \frac{2dz/(1+z^2)}{8/(1+z^2)^2}$$
$$= \int_0^1 (1/4)(1+z^2)dz = (1/4)(z+z^3/3)\Big|_0^1 = 1/3$$

b) The curve $r = 1/(1+\cos\theta)$ is a parabola:

$$r + r\cos\theta = 1 \implies r + x = 1 \implies r^2 = (1-x)^2 \implies y^2 = 1-2x$$

This is the region under $y = \sqrt{1-2x}$ in the first quadrant:

$$A = \int_0^{1/2} \sqrt{1-2x}dx = -(1/3)(1-2x)^{3/2}\Big|_0^{1/2} = 1/3$$

5F. Integration by parts. Reduction formulas

5F-1 a) $\displaystyle\int x^a \ln x dx = \int \ln x d(\frac{x^{a+1}}{a+1}) = \ln x \cdot \frac{x^{a+1}}{a+1} - \int \frac{x^{a+1}}{a+1}\cdot\frac{1}{x}dx$

$$= \frac{x^{a+1}\ln x}{a+1} - \int \frac{x^a}{a+1}dx = \frac{x^{a+1}\ln x}{a+1} - \frac{x^{a+1}}{(a+1)^2} + c \ (a \ne -1)$$

b) $\displaystyle\int x^{-1}\ln x dx = (\ln x)^2/2 + c \ \ (u = \ln x,\ du = dx/x)$

5F-2 a) $\displaystyle\int xe^x dx = \int xd(e^x) = x\cdot e^x - \int e^x dx = x \cdot e^x - e^x + c$

b) $\displaystyle\int x^2 e^x dx = \int x^2 d(e^x) = x^2 \cdot e^x - \int e^x \cdot 2x dx$

$$= x^2 \cdot e^x - 2\int xe^x dx = x^2 \cdot e^x - 2x \cdot e^x + 2e^x + c$$

c) $\int x^3 e^x dx = \int x^3 d(e^x) = x^3 \cdot e^x - \int e^x \cdot 3x^2 dx$

$$= x^3 \cdot e^x - 3\int x^2 e^x dx = x^3 \cdot e^x - 3x^2 \cdot e^x + 6x \cdot e^x - 6e^x + c$$

d) $\int x^n e^{ax} dx = \int x^n d(\frac{e^{ax}}{a}) = \frac{e^{ax}}{a} \cdot x^n - \int \frac{e^{ax}}{a} \cdot nx^{n-1} dx$

$$= \frac{e^{ax}}{a} \cdot x^n - \frac{n}{a}\int x^{n-1} e^{ax} dx$$

5F-3

$$\int \sin^{-1}(4x)dx = x \cdot \sin^{-1}(4x) - \int xd(\sin^{-1}(4x)) = x \cdot \sin^{-1}(4x) - \int x \cdot \frac{4dx}{\sqrt{1-(4x)^2}}$$

$$= x \cdot \sin^{-1}(4x) + \int \frac{du}{8\sqrt{u}} \quad (\text{put } u = 1 - 16x^2,\ du = -32xdx)$$

$$= x \cdot \sin^{-1}(4x) + \frac{1}{4}\sqrt{u} + c$$

$$= x \cdot \sin^{-1}(4x) + \frac{1}{4}\sqrt{1-16x^2} + c$$

5F-4

$$\int e^x \cos x dx = \int e^x d(\sin x) = e^x \sin x - \int e^x \sin x dx$$

$$= e^x \sin x - \int e^x d(-\cos x)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x dx$$

Add $\int e^x \cos x dx$ to both sides to get

$$2\int e^x \cos x dx = e^x \sin x + e^x \cos x + c$$

Divide by 2 and replace the arbitrary constant c by $c/2$:

$$\int e^x \cos x dx = (e^x \sin x + e^x \cos x)/2 + c$$

5F-5

$$\int \cos(\ln x)dx = x \cdot \cos(\ln x) - \int xd(\cos(\ln x))$$

$$= x \cdot \cos(\ln x) + \int \sin(\ln x)dx$$

$$= x \cdot \cos(\ln x) + x \cdot \sin(\ln x) - \int xd(\sin(\ln x))$$

$$= x \cdot \cos(\ln x) + x \cdot \sin(\ln x) - \int \cos(\ln x)dx$$

Add $\int \cos(\ln x)dx$ to both sides to get

$$2\int \cos(\ln x)dx = x\cos(\ln x) + x\sin(\ln x) + c$$

Divide by 2 and replace the arbitrary constant c by $c/2$:

$$\int \cos(\ln x)dx = (x\cos(\ln x) + x\sin(\ln x))/2 + c$$

5F-6 Put $t = e^x \implies dt = e^x dx$ and $x = \ln t$. Therefore

$$\int x^n e^x dx = \int (\ln t)^n dt$$

Integrate by parts:

$$\int (\ln t)^n dt = t \cdot (\ln t)^n - \int t d(\ln t)^n = t(\ln t)^n - n\int (\ln t)^{n-1} dt$$

because $d(\ln t)^n = n(\ln t)^{n-1}t^{-1}dt$.

Unit 6. Additional Topics

6A. Indeterminate forms; L'Hospital's rule

6A-1 a) $\lim_{x\to 0}\frac{\sin 3x}{x} = \lim_{x\to 0}\frac{3\cos 3x}{1} = 3$

b) $\lim_{x\to 0}\frac{\cos(x/2)-1}{x^2} = \lim_{x\to 0}\frac{(-1/2)\sin(x/2)}{2x} = \lim_{x\to 0}\frac{(-1/4)\cos(x/2)}{2} = -1/8$

c) $\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1/x}{1} = 0$

d) $\lim_{x\to 0}\frac{x^2-3x-4}{x+1} = -4.$ Can't use L'Hospital's rule.

e) $\lim_{x\to 0}\frac{\tan^{-1}x}{5x} = \lim_{x\to 0}\frac{1/(1+x^2)}{5} = 1/5$

f) $\lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{1-\cos x}{3x^2} = \lim_{x\to 0}\frac{\sin x}{6x} = \lim_{x\to 0}\frac{\cos x}{6} = 1/6$

g) $\lim_{x\to 1}\frac{x^a-1}{x^b-1} = \lim_{x\to 1}\frac{ax^{a-1}}{bx^{b-1}} = a/b$

h) $\lim_{x\to 1}\frac{\tan(x)}{\sin(3x)} = \frac{\tan 1}{\sin 3}.$ Can't use L'Hospital's rule.

i) $\lim_{x\to\pi}\frac{\ln\sin(x/2)}{x-\pi} = \lim_{x\to\pi}\frac{(1/2)\cot(x/2)}{1} = 0$

j) $\lim_{x\to\pi}\frac{\ln\sin(x/2)}{(x-\pi)^2} = \lim_{x\to\pi}\frac{(1/2)\cot(x/2)}{2(x-\pi)} = \lim_{x\to\pi}\frac{(-1/4)\csc^2(x/2)}{2} = -1/8$

6A-2 a) $x^x = e^{x\ln x} \to e^0 = 1$ as $x\to 0^+$ because

$$\lim_{x\to 0+} x\ln x = \lim_{x\to 0+}\frac{\ln x}{1/x} = \lim_{x\to 0+}\frac{1/x}{-1/x^2} = \lim_{x\to 0+} -x = 0$$

b) $x^{1/x}\to 0$ as $x\to 0^+$ because $x\to 0$ and $1/x\to\infty$.

Slow way using logs:

$x^{1/x} = e^{\frac{\ln x}{x}} \to e^{-\infty} = 0$ as $x\to 0^+$ because

$$\lim_{x\to 0+}\frac{\ln x}{x} = \frac{-\infty}{0^+} = -\infty.$$ (Can't use L'Hospital's rule.)

c) Can't use L'Hospital's rule. Here are two ways:

$(1/x)^{\ln x}\to(\infty)^{-\infty} = 0$ or $(1/x)^{\ln x} = e^{\ln x\ln(1/x)} = e^{-(\ln x)^2}\to e^{-\infty} = 0$

d) $(\cos x)^{1/x} = e^{\frac{\ln\cos x}{x}} \to e^0 = 1$ as $x\to 0^+$ because

$$\lim_{x\to 0+}\frac{\ln\cos x}{x} = \lim_{x\to 0+}\frac{-\tan x}{1} = 0$$

e) $x^{1/x} = e^{\frac{\ln x}{x}} \to e^0 = 1$ as $x\to\infty$ because

$$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1/x}{1} = 0$$

f) $(1+x^2)^{1/x} = e^{\frac{\ln(1+x^2)}{x}} \to e^0 = 1$ as $x \to 0^+$ because

$$\lim_{x\to 0^+} \frac{\ln(1+x^2)}{x} = \lim_{x\to 0^+} \frac{2x/(1+x^2)}{1} = 0$$

g) $(1+3x)^{10/x} = e^{\frac{10\ln(1+3x)}{x}} \to e^{30}$ as $x \to 0^+$ because

$$\lim_{x\to 0^+} \frac{10\ln(1+3x)}{x} = \lim_{x\to 0^+} \frac{10\cdot 3/(1+3x)}{1} = 30$$

h) $\lim\limits_{x\to\infty} \dfrac{x+\cos x}{x} = (?) \lim\limits_{x\to\infty} \dfrac{1-\sin x}{1}$ But the second limit does not exist, so L'Hospital's rule is **inconclusive**. But the first limit does exist after all:

$$\lim_{x\to\infty} \frac{x+\cos x}{x} = \lim_{x\to\infty} 1 + \frac{\cos x}{x} = 1$$

because

$$\frac{|\cos x|}{x} \le \frac{1}{x} \to 0 \quad \text{as } x \to \infty$$

Commentary: L'Hospital's rule does a poor job with oscillatory functions.

i) Fast way: Substitute $u = 1/x$.

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{u\to 0} \frac{\sin u}{u} = \lim_{u\to 0} \frac{\cos u}{1} = 1$$

Slower way:

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{x\to\infty} \frac{(-1/x^2)\cos(1/x)}{-1/x^2} = \cos 0 = 1$$

j) $\left(\dfrac{x}{\sin x}\right)^{1/x^2} = e^{\frac{\ln(x/\sin x)}{x^2}} \to e^{\frac{1}{6}}$ because

$$\lim_{x\to 0^+} \frac{\ln(x/\sin x)}{x^2} = 1/6$$

This is a difficult limit. Although it can be done by L'Hospital's rule the easiest way to work it out is with quadratic (and even cubic!) approximations:

$$\frac{x}{\sin x} \approx \frac{x}{x - x^3/6} = \frac{1}{1 - x^2/6} \approx 1 + x^2/6$$

Hence,

$$\ln(x/\sin x) \approx \ln(1 + x^2/6) \approx x^2/6$$

Therefore,

$$\frac{1}{x^2}\ln(x/\sin x) \to 1/6 \quad \text{as } x \to 0$$

k) Obvious cases: If the exponents are positive (or one 0 and the other positive) then the limit is infinite. If the exponents are both negative (or one 0 and the other negative) then the limit is 0. Also if both exponents are 0 the limit is 1. *(continued →)*

The remaining cases are the ones where a and b have opposite sign. In both cases a wins. In other words, $a < 0$ implies the limit is 0 and $a > 0$ implies the limit is ∞. To show this requires only one use of L'Hospital's rule. For $\alpha > 0$,

$$\lim_{x\to\infty} \frac{x^\alpha}{\ln x} = \lim_{x\to\infty} \frac{\alpha x^{\alpha-1}}{1/x} = \lim_{x\to\infty} \alpha x^\alpha = \infty$$

If $a > 0$ and $b < 0$, let $c = -b > 0$. Then

$$x^a(\ln x)^b = \left(\frac{x^{a/c}}{\ln x}\right)^c \to \infty \quad \text{as } x \to \infty$$

using $\alpha = a/c > 0$. The case $a < 0$ and $b > 0$ is the reciprocal so it tends to 0.

6A-3 Using L'Hospital's rule and $\frac{d}{da}x^{a+1} = x^{a+1}\ln x$,

$$\lim_{a\to-1}\left(\frac{x^{a+1}}{a+1} - \frac{1}{a+1}\right) = \lim_{a\to-1} \frac{x^{a+1}-1}{a+1} = \lim_{a\to-1} \frac{x^{a+1}\ln x}{1} = \ln x$$

6A-4

$$\int_1^x t^a \ln t\,dt = \frac{x^{a+1}\ln x}{a+1} - \frac{x^{a+1}}{(a+1)^2} + \frac{1}{(a+1)^2}$$

Therefore, using L'Hospital's rule and $\frac{d}{da}x^{a+1} = x^{a+1}\ln x$,

$$\begin{aligned}\lim_{a\to-1}\int_1^x t^a \ln t\,dt &= \lim_{a\to-1} \frac{(a+1)x^{a+1}\ln x - x^{a+1} + 1}{(a+1)^2} \\ &= \lim_{a\to-1} \frac{(a+1)x^{a+1}(\ln x)^2}{2(a+1)} \\ &= (\ln x)^2/2 = \int_1^x t^{-1}\ln t\,dt\end{aligned}$$

6A-5 You can't use L'Hospital's rule for $\lim_{x\to0} \frac{6x-4}{2-2x}$ because the nominator and denominator are not going to zero as $x \to 0$. The first equality is true, but the second one is false.

6A-6 a) $y = xe^{-x}$ is defined on $-\infty < x < \infty$.

$$y' = (1-x)e^{-x} \text{ and } y'' = (-2+x)e^{-x}$$

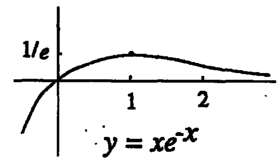
Therefore, $y' > 0$ for $x < 1$ and $y' < 0$ for $x > 1$; $y'' > 0$ for $x > 2$ and $y'' < 0$ for $x < 2$.

Endpoint values: $y \to -\infty$ as $x \to -\infty$, because $e^{-x} \to \infty$ as $x \to -\infty$. By L'Hospital's rule,

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

Critical value: $y(1) = 1/e$.

Graph: $(-\infty,-\infty) \nearrow (1, 1/e) \searrow (\infty, 0)$.



Concave up on: $2 < x < \infty$, concave down on: $-\infty < x < 2$.

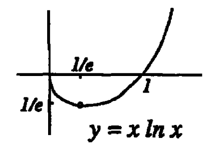
b) $y = x \ln x$ is defined on $0 < x < \infty$.

$$y' = \ln x + 1, \quad y'' = 1/x$$

Therefore, $y' > 0$ for $x > 1/e$ and $y' < 0$ for $x < 1/e$; $y'' > 0$ for all $x > 0$.

Endpoint values: As $x \to \infty$, both x and $\ln x$ tend to infinity, so $y \to \infty$. By L'Hospital's rule,

$$\lim_{x \to 0+} x \ln x = \lim_{x \to 0+} \frac{\ln x}{x} = \lim_{x \to 0+} \frac{1/x}{1} = 0$$



Critical value: $y(1/e) = -1/e$.

Graph: $(0,0) \searrow (1/e, -1/e) \nearrow (\infty, \infty)$, crossing zero at $x = e$. Concave up for all $x > 0$.

c) $y = x/\ln x$ is defined on $0 < x < \infty$, except for $x = 1$.

$$y' = \frac{\ln x - 1}{(\ln x)^2}$$

Thus, $y' < 0$ for $0 < x < 1$ and for $1 < x < e$ and $y' > 0$ for $x > e$;

Endpoint values: $y \to 0$ as $x \to 0^+$ because $x \to 0$ and $1/\ln x \to 0$. L'Hôpital's rule implies

$$\lim_{x \to \infty} \frac{x}{\ln x} = \lim_{x \to \infty} \frac{1}{1/x} = \infty$$

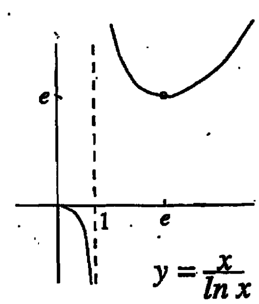
Singular values: $y(1^+) = \infty$ and $y(1^-) = -\infty$.

Critical value: $y(e) = e$.

Graph: $(0,0) \searrow (1, -\infty) \uparrow (1, \infty) \searrow (e, e) \nearrow (\infty, \infty)$.

To determine where it is convex and concave:

$$y'' = \frac{2 - \ln x}{x(\ln x)^3}$$



We have $y'' = 0$ when $\ln x = 2$, i.e., when $x = e^2$. From this,

$y'' < 0$ for $0 < x < 1$ and for $x > e^2$ and $y'' > 0$ for $1 < x < e^2$.

Concave (down) on: $0 < x < 1$ and $x > e^2$

Convex (concave up) on: $1 < x < e^2$

Inflection point: $(e^2, e^2/2)$ (too far to the right to show on the graph)

6B. Improper integrals

6B-1 $\dfrac{dx}{\sqrt{x^3+5}} < \dfrac{1}{\sqrt{x^3}}$ for $x > 0$

$\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^3+5}} < \int_1^\infty \frac{dx}{x^{3/2}}$ which converges, by INT (4)

Answer: converges

6B-2 $\dfrac{x^2dx}{x^3+2} \simeq \dfrac{1}{x}$ if $x >> 1$, so we guess divergence.

$\dfrac{x^2dx}{x^3+2} > \dfrac{1}{2x}$ if $2x^3 > x^3+2$ or $x^3 > 2$ or $x > 2^{1/3}$

$\displaystyle\int_2^\infty \frac{x^2dx}{x^3+2} > \frac{1}{2}\int_2^\infty \frac{dx}{x}$, which diverges by INT (4).

$\displaystyle\int_2^\infty \frac{x^2dx}{x^3+2}$ diverges, by comp.test, and so does $\displaystyle\int_0^\infty \frac{x^2dx}{x^3+2}$ by INT (3).

6B-3 $\displaystyle\int_0^1 \frac{dx}{x^3+x^2}$ integrand blows up at $x=0$

$\dfrac{1}{x^3+x^2} = \dfrac{1}{x^2(x+1)} \sim \dfrac{1}{x^2}$ when $x \simeq 0$

So we guess divergence.

$\dfrac{1}{x^3+x^2} > \dfrac{1}{2x^2}$ if $2x^2 > x^3+x^2$ or $x^2 > x^3$; true if $0 < x < 1$.

$\displaystyle\Longrightarrow \int_0^1 \frac{dx}{x^3+x^2} > \frac{1}{2}\int_0^1 \frac{dx}{x^2}$ which diverges by INT (6)

6B-4 $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ blows up at $x=1$

$\dfrac{1}{\sqrt{1-x^3}} = \dfrac{1}{\sqrt{(1-x)(1+x+x^2)}} \sim \dfrac{1}{\sqrt{3}\sqrt{1-x}}$ for $x \simeq 1$

So we guess convergence.

$\dfrac{1}{\sqrt{1-x^3}} < \dfrac{1}{\sqrt{1-x}}$ if $x^3 < x$ OK if $0 < x < 1$

$\dfrac{1}{\sqrt{1-x}}$ converges by INT (6), so $\dfrac{1}{\sqrt{1-x^3}}$ also converges by comp.test.

6B-5 $\displaystyle\int_0^\infty \frac{e^{-x}dx}{x}$ is improper at both ends.

At the ∞ end it converges, since

$\dfrac{e^{-x}dx}{x} < e^{-x}$ if $x > 1$ and $\displaystyle\int_0^\infty e^{-x}$converges.

At the 0 end: trouble! $\dfrac{e^{-x}dx}{x} \sim \dfrac{1}{x}$. So we guess divergence.

$\dfrac{e^{-x}dx}{x} > \dfrac{1}{4x}$ on $0 < x < 1 \Longrightarrow \int_0^\infty \dfrac{e^{-x}dx}{x} > \dfrac{1}{4}\int_0^\infty \dfrac{dx}{x}$ divergent.

$\Longrightarrow \int_0^\infty \dfrac{e^{-x}dx}{x}$ diverges —one end is infinite (the 0 end!)

6B-6 $\displaystyle\int_1^\infty \frac{\ln x dx}{x^2}$

Here $\ln x$ grows so slowly, that we suspect convergence.

$\dfrac{\ln x}{x^2} < \dfrac{x}{x^2}$ is not convergent.

How about $\dfrac{\ln x}{x^2} < \dfrac{1}{x^{3/2}}$? if $x >> 1$. This says $\dfrac{\ln x}{\sqrt{x}} < 1$ if $x >> 1$ and this is true, since

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/2\sqrt{x}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

$\Longrightarrow \int_1^\infty \dfrac{\ln x dx}{x^2} < \dfrac{x}{x^{3/2}}$ converges, by INT (4).

So $\int_1^\infty \dfrac{\ln x dx}{x^2}$ converges by comp.test.

These have been written out in detail, to review the reasoning. Your own solutions don't have to be so detailed.

6B-7 a) $\displaystyle\int_0^\infty e^{-8x}dx = -(1/8)e^{-8x}\Big|_0^\infty = 1/8$ convergent

b) $\displaystyle\int_1^\infty x^{-n}dx = \frac{x^{-n+1}}{-n+1}\Big|_1^\infty = \frac{1}{n-1}$ convergent ($n > 1$)

c) divergent

d) $\displaystyle\int_0^2 \frac{xdx}{\sqrt{4-x^2}} = -(4-x^2)^{1/2}\Big|_0^2 = 2$ convergent

e) $\displaystyle\int_0^2 \frac{dx}{\sqrt{2-x}} = -2(2-x)^{1/2}\Big|_0^2 = 2\sqrt{2}$ convergent

f) $\displaystyle\int_e^\infty \frac{dx}{x(\ln x)^2} = -(\ln x)^{-1}\Big|_e^\infty = 1$ convergent

g) $\displaystyle\int_0^1 \frac{dx}{x^{1/3}} = (3/2)x^{2/3}\Big|_0^1 = \frac{3}{2}$ convergent

h) divergent (at $x = 0$)

i) divergent (at $x = 0$)

j) Convergent because $\ln x$ tends to $-\infty$ more slowly than any power as $x \to 0^+$.

Integrate by parts

$$\int_0^1 \ln x dx = x \ln x - x|_0^1 = -1$$

(Need L'Hospital's rule to check that $x \ln x \to 0$ as $x \to 0^+$.)

k) Convergent because $|e^{-2x} \cos x| < e^{-2x}$. Evaluate by integrating by parts twice (as in E30/4).

$$\int_0^\infty e^{-2x} \cos x dx = \frac{1}{5} e^{-2x} \sin x - \frac{2}{5} e^{-2x} \cos x \Big|_0^\infty = 2/5$$

l) divergent ($\int_e^\infty \frac{dx}{x \ln x} = \ln \ln x|_e^\infty = \infty$)

m) $\int_0^\infty \frac{dx}{(x+2)^3} = (-1/2)(x+2)^{-2}|_0^\infty = 1/8$ convergent

n) divergent (at $x = 2$)

o) divergent (at $x = 0$)

p) divergent (at $x = \pi/2$)

6B-8 a) $\lim_{x\to\infty} \frac{\int_0^x e^{t^2} dt}{e^{x^2}} = \lim_{x\to\infty} \frac{e^{x^2}}{2xe^{x^2}} = \lim_{x\to\infty} \frac{1}{2x} = 0$ (L'Hospital and FT2)

b) $\lim_{x\to\infty} \frac{\int_0^x e^{t^2} dt}{e^{x^2}/x} = \lim_{x\to\infty} \frac{e^{x^2}}{2x^2 e^{x^2} - e^{x^2}/x^2} = \lim_{x\to\infty} \frac{1}{2 - (1/x^2)} = \frac{1}{2}$

c) $\lim_{x\to\infty} \int_0^x e^{-t^2} dt = A$ a finite number > 0 because the integral is convergent. But $e^{x^2} \to \infty$, so the whole limit tends to infinity.

d) $= \lim_{a\to 0+} \frac{\int_a^1 x^{-1/2} dx}{1/\sqrt{a}} = \lim_{a\to 0+} \frac{-1/\sqrt{a}}{(-1/2)a^{-3/2}} = \lim_{a\to 0+} 2a = 0$ (L'Hospital and FT2)

e) $= \lim_{a\to 0+} \frac{\int_a^1 x^{-3/2} dx}{1/\sqrt{a}} = \lim_{a\to 0+} \frac{-a^{-3/2}}{(-1/2)a^{-3/2}} = 2$ (L'Hospital and FT2)

(f)

$$\begin{aligned}
\lim_{b\to(\pi/2)+} (b - \pi/2) \int_0^b \frac{dx}{1 - \sin x} &= \lim_{b\to(\pi/2)+} \frac{\int_0^b \frac{dx}{1-\sin x}}{1/(b - \pi/2)} \\
&= \lim_{b\to(\pi/2)+} \frac{1/(1 - \sin b)}{-1/(b - \pi/2)^2} \\
&= \lim_{b\to(\pi/2)+} \frac{(b - \pi/2)^2}{\sin b - 1} \\
&= \lim_{b\to(\pi/2)+} \frac{2(b - \pi/2)}{\cos b} \\
&= \lim_{b\to(\pi/2)+} \frac{2}{-\sin b} = -2
\end{aligned}$$

6C. Infinite Series

6C-1 a) $1+\frac{1}{5}+\frac{1}{25}+\cdots=1+\frac{1}{5}+\frac{1}{5^2}+\cdots=\frac{1}{1-\frac{1}{5}}=\frac{5}{4}$

b) $8+2+\frac{1}{2}+\cdots=8(1+\frac{1}{4}+\frac{1}{4^2}+\cdots)=8(\frac{1}{1-\frac{1}{4}})=\frac{6B}{3}$

c) $\frac{1}{4}+\frac{1}{5}+\cdots=\frac{1}{4}(1+\frac{4}{5}+(\frac{4}{5})^2+\cdot)=\frac{1}{4}(\frac{1}{1-\frac{4}{5}})=\frac{5}{4}$

d) $0.4444\cdots=0.4(1+0.1+0.1^2+0.1^3+\cdots)=0.4(\frac{1}{1-0.1})=0.4(\frac{1}{0.9})=\frac{4}{9}$

e)$0.0602602602\cdots=0.0602(1+0.001+0.000001+\cdots)=0.0602(\frac{1}{1-0.001})$

$=\frac{0.0602}{0.999}=\frac{301}{4995}$

6C-2 a) $1+1/2+1/3+1/4+\cdots$

clearly, we have $1>\int_1^2\frac{1}{x}dx, \frac{1}{2}>\int_2^3\frac{1}{x}dx,\cdots$

so we will have $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots>\int_1^2\frac{1}{x}dx+\int_2^3\frac{1}{x}dx+\int_3^4\frac{1}{x}dx+\int_4^5\frac{1}{x}dx+\cdots=$ $\int_1^\infty\frac{1}{x}dx$, which is divergent, so the infinite series is divergent.

b) $\sum_{n=1}^{\infty}\frac{1}{n^p}$

Case 1: $p\le 1$. $\frac{1}{n^p}>\int_n^{n+1}\frac{dx}{x^p}$

$\Longrightarrow\sum_{n=1}^{\infty}\frac{1}{n^p}>\int_1^\infty\frac{dx}{x^p}$, which is divergent, so the infinite series is divergent.

Case 2: $p>1$

$\frac{1}{n^p}<\int_{n-1}^{n}\frac{dx}{x^p}\Longrightarrow\sum_{n=1}^{\infty}\frac{1}{n^p}<1+\int_1^\infty\frac{dx}{x^p}$, which is convergent. So the infinite series is convergent.

c) $1/2+1/4+1/6+1/8+\cdots=(1/2)(1+1/2+1/3+1/4+\cdots)$. So from a), the series is divergent.

d) $1+1/3+1/5+1/7+\cdots$

$1>1/2, 1/3>1/4, 1/5>1/6, 1/7>1/8,\cdots$

So $1+1/3+1/5+1/7+\cdots>1/2+1/4+1/6+1/8+\cdots$ which is divergent from c) Thus the series diverges.

e)
$$\begin{aligned} 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots &= (1-\frac{1}{2})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{5}-\frac{1}{6})+\cdots \\ &= \frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots \\ &< \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots \\ &< \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots \end{aligned}$$

which is convergent by b). So the infinite series is convergent.

f) $n/n! = 1/(n-1)! < 1/(n-1)(n-2) \simeq 1/n^2$ for $n >> 1$. So convergent by comparison with b).

g) Geometric series with ratio $(\sqrt{5}-1)/2 < 1$, so the series is convergent.

h) Geometric series with ratio $(\sqrt{5}+1)(2\sqrt{5}) < 1$, so the series is convergent.

i) Larger than $\sum 1/n$ for $n \geq 3$, so divergent by part b).

j) $\ln n$ grows more slowly than any power. For instance,

$$\ln n < n^{1/2} \implies \frac{\ln n}{n^2} < n^{-3/2} \quad \text{for } n >> 1$$

The series $\sum n^{-3/2}$ converges by part b), so this series also converges.

k) Converges because $\dfrac{n+2}{n^4-5} \simeq \dfrac{1}{n^3}$, and $\sum n^{-3}$ converges by part b).

l) $\dfrac{(n+2)^{1/3}}{(n^4+5)^{1/3}} \simeq \dfrac{n^{1/3}}{n^{4/3}} \simeq \dfrac{1}{n}$. Therefore this series diverges by comparison with $\sum 1/n$.

m) Quadratic approximation implies $\cos(1/n) \approx 1 - 1/2n^2$ and hence

$$\ln(\cos\frac{1}{n}) \simeq -1/2n^2 \quad \text{as } n \to \infty$$

Hence the series converges by comparison with $\sum 1/n^2$ from part b).

n) e^{-n} beats n^2 by a large margin. For example, L'Hospital's rule implies

$$e^{-n/2}n^2 \to 0 \quad \text{as } n \to \infty$$

Therefore for large n, $n^2e^{-n} = n^2e^{-n/2}e^{-n/2} < e^{-n/2}$ and $\sum e^{-n/2}$ is a convergent geometric series. Therefore the original series converges by comparison.

o) Just as in part (n), $e^{-\sqrt{n}}$ beats n^2 by a large margin. L'Hospital's rule implies

$$e^{-m/2}m^4 \to 0 \quad \text{as } m \to \infty$$

Put $m = \sqrt{n}$ to get

$$e^{-\sqrt{n}/2}n^2 \to 0 \quad \text{as } n \to \infty$$

Therefore for large n, $n^2e^{-\sqrt{n}} = n^2e^{-\sqrt{n}/2}e^{-\sqrt{n}/2} < e^{-\sqrt{n}/2}$. Moreover, we also have

$$e^{-\sqrt{n}} < 1/n^2 \quad n \text{ large}$$

Thus the sum is dominated by $\sum e^{-\sqrt{n}/2} < \sum 1/n^2$ and is convergent by comparison with part b).

6C-3 a)

$$\ln n = \int_1^n \frac{dx}{x} < \text{ Upper sum } = 1 + \frac{1}{2} + \cdots \frac{1}{n-1} < 1 + \frac{1}{2} + \cdots \frac{1}{n}$$

In other words,

$$\ln n < 1 + \frac{1}{2} + \cdots \frac{1}{n}$$

On the other hand,

$$\ln n = \int_1^n \frac{dx}{x} > \text{ Lower sum } = \frac{1}{2} + \cdots \frac{1}{n}$$

Adding 1 to both sides,

$$1 + \ln n > 1 + \frac{1}{2} + \cdots \frac{1}{n}$$

b) Need at least $\ln n = 999$

$$\text{Time} > 10^{-10}e^{999} \approx 7 \times 10^{423} \text{ seconds}$$

This is far, far longer than the estimated time from the "big bang."

Unit 7. Infinite Series

7A: Basic Definitions

7A-1

a) Sum the geometric series: $\sum_0^\infty \frac{1}{4^n} = \sum_0^\infty \left(\frac{1}{4}\right)^n = \frac{1}{1-(1/4)} = \frac{4}{3}$.

b) $1 - 1 + 1 - 1 + \ldots + (-1)^n + \ldots$ diverges, since the partial sums s_n are successively $1, 0, 1, 0, \ldots$, and therefore do not approach a limit.

c) Diverges, since the n-th term $\frac{n-1}{n}$ does not tend to 0 (using the n-th term test for divergence).

d) The given series $= \ln 2 + \frac{1}{2}\ln 2 + \frac{1}{3}\ln 2 + \ldots = \ln 2(1 + \frac{1}{2} + \frac{1}{3} + \ldots)$; but $\sum_1^\infty 1/n$ diverges; therefore the given series diverges.

e) $\sum_1^\infty \frac{2^{n-1}}{3^n} = \frac{1}{3}\sum_1^\infty \frac{2^{n-1}}{3^{n-1}}$, geometric series with sum $\frac{1}{3}\left(\frac{1}{1-(2/3)}\right) = \frac{1}{3}\cdot 3 = 1$.

f) series $= \sum_0^\infty \left(\frac{-1}{3}\right)^n = \frac{1}{1-(-1/3)} = \frac{3}{4}$ (sum of a geometric series)

7A-2 $.21111\ldots = .2 + .01 + .001 + \ldots = .2 + .01(1 + \frac{1}{10} + \frac{1}{10^2} + \ldots = .2 + .01(\left(\frac{1}{1-1/10}\right) = \frac{19}{90}$.

7A-3 Geometric series; converges if $|x/2| < 1$, i.e., if $|x| < 2$, or equivalently, $-2 < x < 2$.

7A-4

a) Partial sum: $s_m = \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \ldots + \left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{m+1}}\right)$

$$= 1 - \frac{1}{\sqrt{m+1}} \to 1 \text{ as } m \to \infty. \text{ Therefore the sum is 1.}$$

b) $\frac{1}{n(n+2)} = \frac{1/2}{n} + \frac{-1/2}{n+2}$; therefore $\sum_1^\infty \frac{1}{n(n+2)} = \frac{1}{2}\left(\sum_0^\infty \left(\frac{1}{n} - \frac{1}{n+2}\right)\right)$.

The m-th partial sum of the series is

$$s_m = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \ldots + \frac{1}{m} - \frac{1}{m+2}\right) = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{m+1} - \frac{1}{m+2}\right),$$

since all other terms cancel.

Therefore $s_m \to \frac{3}{4}$ as $m \to \infty$, so the sum is 3/4.

7A-5 The distance the ball travels is $h + \frac{2}{3}h + \frac{2}{3}h + \frac{2}{3}\left(\frac{2}{3}h\right) + \frac{2}{3}\left(\frac{2}{3}h\right) + \ldots$; the successive terms give the first down, the first up, the second down, and so on. Add h to the series to make the terms uniform; you get a geometric series to sum:

$$2\left(h + 2h/3 + (2/3)^2 h + \ldots\right) = 2h(1 + 2/3 + (2/3)^2 + \ldots) = 2h\left(\frac{1}{1-2/3}\right) = 6h.$$

Subtracting the h that we added on gives: the total distance traveled $= 5h$.

7B: Convergence Tests

7B-1

a) $\displaystyle\int_0^\infty \frac{x}{x^2+4} = \frac{1}{2}\ln(x^2+4)\Big]_0^\infty = \infty$; divergent

b) $\displaystyle\int_0^\infty \frac{1}{x^2+1} = \tan^{-1}x\Big]_0^\infty = \frac{\pi}{2}$; convergent

c) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x+1}} = 2(x+1)^{1/2}\Big]_0^\infty = \infty$; divergent

d) $\displaystyle\int_1^\infty \frac{\ln x}{x} = \frac{1}{2}(\ln x)^2\Big]_1^\infty = \infty$; divergent

e) $\displaystyle\int_2^\infty \frac{1}{(\ln x)^p\cdot x} = \frac{(\ln x)^{1-p}}{1-p}\Big]_2^\infty$, if $p \neq 1$: divergent if $p < 1$, convergent if $p > 1$

If $p = 1$, $\displaystyle\int_2^\infty \frac{dx}{\ln x} = \ln(\ln x)\Big]_2^\infty = \infty$. Thus series converges if $p > 1$, diverges if $p \le 1$.

f) $\displaystyle\int_1^\infty \frac{1}{x^p} = \frac{x^{1-p}}{1-p}\Big]_1^\infty$, if $p \neq 1$; diverges if $p < 1$, converges if $p > 1$.

If $p = 1$, $\displaystyle\int_1^\infty \frac{dx}{x} = \ln x\Big]_1^\infty = \infty$; thus series converges if $p > 1$, diverges if $p \le 1$.

7B-2

a) Convergent; compare with $\displaystyle\sum_1^\infty \frac{1}{n^2}$: $\displaystyle\frac{n^2}{n^2+3n} = \frac{1}{1+3/n} \to 1$ as $n \to \infty$

b) Divergent; compare with $\displaystyle\sum \frac{1}{n}$: $\displaystyle\frac{n}{n+\sqrt{n}} = \frac{1}{1+1/\sqrt{n}} \to 1$, as $n \to \infty$

c) Divergent; compare with $\displaystyle\sum \frac{1}{n}$: $\displaystyle\frac{n}{\sqrt{n^2+n}} = \frac{1}{\sqrt{1+1/n}} \to 1$, as $n \to \infty$

d) Convergent; compare with $\displaystyle\sum_1^\infty \frac{1}{n^2}$: $\displaystyle\lim_{n\to\infty} n^2\sin\left(\frac{1}{n^2}\right) = \lim_{h\to 0}\frac{\sin h}{h} = 1$

e) Convergent; compare with $\displaystyle\sum_1^\infty \frac{1}{n^{3/2}}$: $\displaystyle\frac{n^{3/2}\sqrt{n}}{n^2+1} = \frac{n^2}{n^2+1} = \frac{1}{1+1/n^2} \to 1$ as $n \to \infty$

f) Divergent, by comparison test : $\displaystyle\frac{\ln n}{n} > \frac{1}{n}$; $\displaystyle\sum_1^\infty \frac{1}{n}$ diverges

g) Convergent; compare with $\displaystyle\sum \frac{1}{n^2}$: $\displaystyle\frac{n^2\cdot n^2}{n^4-1} = \frac{n^4}{n^4-1} \to 1$ as $n \to \infty$

h) Divergent; compare with $\displaystyle\sum \frac{1}{4n}$: $\displaystyle\frac{4n\cdot n^3}{4n^4+n^2} = \frac{1}{1+1/4n^2} \to 1$

7B-3 By the mean-value theorem, $\sin x < x$, if $x > 0$; therefore $\sum_0^\infty \sin a_n < \sum_0^\infty a_n$; so the series converges by the comparison test.

7B-4

a) By ratio test, $\dfrac{n+1}{2^{n+1}} \cdot \dfrac{2^n}{n} = \left(\dfrac{n+1}{n}\right) \cdot \dfrac{1}{2} \to \dfrac{1}{2}$ as $n \to \infty$; convergent

b) By ratio test, $\dfrac{2^{n+1}}{(n+1)!} \cdot \dfrac{n!}{2^n} = \dfrac{2}{n+1} \to 0$ as $n \to \infty$; convergent

c) By ratio test, $\dfrac{2^{n+1}}{1 \cdot 3 \cdot \dots \cdot 2n+1} \cdot \dfrac{1 \cdot 3 \cdot \dots \cdot 2n-1}{2^n} = \dfrac{2}{2n+1} \to 0$ as $n \to \infty$; convergent

d) By ratio test, $\dfrac{(n+1)!^2}{(2n+2)!} \cdot \dfrac{(2n)!}{n!^2} = \dfrac{(n+1)^2}{(2n+2)(2n+1)} \to \dfrac{1}{4}$ as $n \to \infty$; convergent

e) Ratio test fails: $\dfrac{1}{\sqrt{n+1}} \cdot \dfrac{\sqrt{n}}{1} \to 1$ as $n \to \infty$; but $\sum \dfrac{1}{\sqrt{n}}$ diverges; therefore the series is not absolutely convergent.

f) By ratio test, $\dfrac{(n+1)!}{(n+1)^{n+1}} \cdot \dfrac{n^n}{n!} = \dfrac{n^n}{(n+1)^n} = \dfrac{1}{(1+1/n)^n} \to \dfrac{1}{e} < 1$ as $n \to \infty$; convergent

g) Ratio test fails: $\dfrac{1}{(n+1)^2} \cdot \dfrac{n^2}{1} \to 1$ as $n \to \infty$; but $\sum \dfrac{1}{n^2}$ converges; therefore the series is absolutely convergent.

h) Ratio test fails: $\sum \dfrac{1}{\sqrt{n^2+1}}$ diverges, by limit comparison with $\sum \dfrac{1}{n}$; therefore the series is not absolutely convergent.

i) Ratio test fails: $\sum \dfrac{n}{n+1}$ diverges by the n-th term test; therefore the series is not absolutely convergent

7B-5

e) conditionally convergent: terms alternate in sign, $\dfrac{1}{\sqrt{n}} \to 0$, decreasing;

h) conditionally convergent: terms alternate in sign, $\dfrac{1}{\sqrt{n^2+1}} \to 0$, decreasing;

i) divergent, by the n-th term test: $\lim\limits_{n\to\infty} \dfrac{(-1)^n n}{n+1} \neq 0$.

7B-6 In all of these, we are using the ratio test.

a) $\dfrac{|x|^{n+1}}{n+1} \cdot \dfrac{n}{|x|^n} = |x| \cdot \left(\dfrac{n}{n+1}\right) \to |x|$ as $n \to \infty$; converges for $|x| < 1$; $R = 1$

b) $\dfrac{2^{n+1}|x|^{n+1}}{(n+1)^2} \cdot \dfrac{n^2}{2^n|x|^n} = 2|x| \cdot \left(\dfrac{n}{n+1}\right)^2 \to 2|x|$ as $n \to \infty$; converges for $2|x| < 1$ or $|x| < 1/2$; $R = 1/2$

c) $\dfrac{(n+1)!|x|^{n+1}}{n!|x|^n} = (n+1)|x| \to \infty$ as $n \to \infty$; converges only for $|x| = 0$; $R = 0$

d) $\dfrac{|x|^{2(n+1)}}{3^{n+1}} \cdot \dfrac{3^n}{|x|^{2n}} = \dfrac{|x|^2}{3} \to \dfrac{|x|^2}{3}$ as $n \to \infty$; converges for $\dfrac{|x|^2}{3} < 1$,
that is, for $|x| < \sqrt{3}$; $R = \sqrt{3}$

e) $\dfrac{|x|^{2n+3}}{2^{n+1}\sqrt{n+1}} \cdot \dfrac{2^n\sqrt{n}}{|x|^{2n+1}} = \dfrac{|x|^2}{2} \cdot \sqrt{\dfrac{n}{n+1}} \to \dfrac{|x|^2}{2}$ as $n \to \infty$; converges for $\dfrac{|x|^2}{2} < 1$ or $|x| < \sqrt{2}$; $R = \sqrt{2}$

f) $\dfrac{(2n+2)!|x|^{2n+2}}{(n+1)!^2} \cdot \dfrac{n!^2}{(2n)!|x|^{2n}} = |x|^2 \cdot \dfrac{(2n+2)(2n+1)}{(n+1)^2} \to 4|x|^2$ as $n \to \infty$;
converges for $4|x|^2 < 1$, or $|x| < 1/2$; $R = 1/2$

g) $\dfrac{|x|^{n+1}}{\ln(n+1)} \cdot \dfrac{\ln n}{|x|^n} = |x| \cdot \dfrac{\ln n}{\ln(n+1)} \to |x|$ as $n \to \infty$; converges for $|x| < 1$; $R = 1$

(By L'Hospital's rule, $\lim\limits_{x\to\infty} \dfrac{\ln x}{\ln(x+1)} = \lim\limits_{x\to\infty} \dfrac{1/x}{1/(x+1)} = 1$.)

h) $\dfrac{2^{2n+2}|x|^{n+1}}{(n+1)!} \cdot \dfrac{n!}{2^{2n}|x|^n} = \dfrac{2^2|x|}{n+1} \to 0$ as $n \to \infty$; converges for all x; $R = \infty$

7C: Taylor Approximations and Series

7C-1

(a)

$y = \cos x$	$y' = -\sin x$	$y'' = -\cos x$	$y^{(3)} = \sin x$	$y^{(4)} = \cos x, \ldots$
$y(0) = 1$	$y'(0) = 0$	$y''(0) = -1$	$y^{(3)}(0) = 0$	$y^{(4)}(0) = 1, \ldots$
$a_0 = 1$	$a_1 = 0$	$a_2 = -1/2!$	$a_3 = 0$	$a_4 = 1/4! \ldots$

The pattern then repeats with the higher coefficients, so we get finally

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots$$

(b)

$y = \ln(1+x)$	$y' = (1+x)^{-1}$	$y'' = -(1+x)^{-2}$	$y^{(3)} = 2!(1+x)^{-3}$	$y^{(4)} = -3!(1+x)^{-4}, \ldots$
$y(0) = 0$	$y'(0) = 1$	$y''(0) = -1$	$y^{(3)}(0) = 2!$	$y^{(4)}(0) = -3!, \ldots$
$a_0 = 0$	$a_1 = 1$	$a_2 = -1/2$	$a_3 = 1/3$	$a_4 = -1/4 \ldots$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n-1}x^n}{n} + \ldots$$

(c) Typical terms in the calculation are given.

$$y = (1+x)^{1/2} \qquad y'' = \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)(1+x)^{-3/2} \qquad y^{(4)} = \frac{(-1)(-3)(-5)}{2^4}(1+x)^{-7/2}$$

$$y(0) = 1 \qquad y''(0) = \frac{-1}{2^2} \qquad y^{(4)}(0) = \frac{(-1)^3(1\cdot 3\cdot 5)}{2^4}$$

$$a_0 = 1 \qquad a_2 = -1/8 \qquad a_4 = -\frac{1\cdot 3\cdot 5}{2^4 4!}$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \ldots + (-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n\cdot n!}x^n + \ldots$$

One gets the same answer by using the binomial formula; this is the way to remember the series:

$$(1+x)^{1/2} = 1 + \left(\frac{1}{2}\right)x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}x^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \ldots$$

7C-2 $\sin x \; = \; x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + R_6(x).$

(We could use either $R_5(x)$ or $R_6(x)$, since the above polynomial is both $T_5(x)$ and $T_6(x)$, but $R_6(x)$ gives a smaller error estimation if $|x| < 1$, since it contains a higher power of x.)

$$R_6(1) \;=\; \frac{\sin^{(7)} c}{7!}\cdot 1^7 \;=\; \frac{-\cos c}{7!}, \text{ for some } 0 < c < 1. \text{ Therefore}$$

$$|R_6(1) \;\le \frac{1}{7!} \;= \frac{1}{5040} \;< .0002$$

Thus $\sin 1 \;\approx\; 1 - \dfrac{1}{3!} + \dfrac{1}{5!} \;\approx\; .84166$; the true value is $\sin 1 = .84147$, which is within the error predicted by the Taylor remainder.

7C-3 Since $f(x) = e^x$, the n-th remainder term is given by

$$R_n(1) \;=\; \frac{f^{(n+1)}(c)}{(n+1)!}\cdot 1^{n+1} \;=\; \frac{e^c}{(n+1)!} \;<\; \frac{3}{(n+1)!} < \frac{5}{10^5} \quad \text{if } n+1 = 8.$$

Therefore we want $n = 7$, i.e., we should use the Taylor polynomial of degree 7; calculation gives $e \approx 1+1+1/2+1/6+1/24+1/120+1/720+1/5040 = 2.71825\ldots$, which is indeed correct to 3 decimal places.

7C-4 Using as in 7C-2 the remainder $R_3(x)$, rather than $R_2(x)$, we have

$$|R_3(x)| \;=\; \left|\frac{\cos^{(4)}(c)}{4!}x^4\right| \;=\; \left|\frac{\cos c}{4!}x^4\right| \le \frac{|x|^4}{4!} \le \frac{(.5)^4}{24} \;=\; .0026.$$

So the answer is no, if $|x| < .5$. (If the interval is shrunk to $|x| < .3$, the answer will be yes, since $(.3)^4/24 < .001$.)

7C-5 By Taylor's formula for e^x, substituting $-x^2$ for x,

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} + \frac{e^c(-x^2)^3}{3!}, \quad 0 < c < .5$$

Since $0 < e^c < 2$, the remainder term is $< \frac{x^6}{3}$; integrating,

$$\int_0^{.5} e^{-x^2}dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10}\right]_0^{.5} + \text{ error } = .461 + \text{error};$$

where $|\text{error}| < \displaystyle\int_0^{.5} \frac{x^6}{3} = \frac{x^7}{21}\Big]_0^{.5} = .00028 < .0003$; thus the answer .461 is good to 3 decimal places.

7D: Power Series

7D-1

(a) $$e^{-2x} = 1 - 2x + \frac{2^2}{2!}x^2 + \ldots + (-1)^n\frac{2^n}{n!}x^n + \ldots,$$

by substituting $-2x$ for x in the series for e^x.

(b) $$\cos\sqrt{x} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \ldots + \frac{(-1)^n x^n}{(2n)!} + \ldots$$

(c) $$\begin{aligned}\sin^2 x &= \frac{1}{2}(1-\cos 2x) = \frac{1}{2}\left(1 - \left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \ldots\right]\right)\\ &= \frac{1}{2}\left(\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \ldots + \frac{(-1)^{n-1}(2x)^{2n}}{(2n)!} + \ldots\right)\end{aligned}$$

(d) Write the series for $1/(1+x)$, differentiate and multiply both sides by -1:

$$\begin{aligned}\frac{1}{1+x} &= 1 - x + x^2 - x^3 + \ldots + (-1)^{n+1}x^{n+1} + \ldots\\ \frac{1}{(1+x)^2} &= 1 - 2x + 3x^2 + \ldots + (-1)^n(n+1)x^n + \ldots\end{aligned}$$

(e) $$D\tan^{-1}x = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots + (-1)^n x^{2n} + \ldots,$$

by substituting x^2 for x in the series for $1/(1+x)$; (cf. (d) above). Now integrate both sides of the above equation:

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + \frac{(-1)^n x^{2n+1}}{2n+1} + \ldots + C;$$

Evaluate the constant of integration by putting $x = 0$, one gets $0 = 0 + C$, so $C = 0$.

(f)
$$D\ln(1+x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots + (-1)^{n+1}x^{n+1} + \ldots$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^n x^{n+1}}{n+1} + \ldots + C,$$

by integrating both sides. Find C by putting $x = 0$, one gets $C = 0$.

(g)
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$
$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Adding and dividing by 2 gives: $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots$

7D-2

a) $$\frac{1}{x+9} = \frac{1/9}{1+x/9} = \frac{1}{9}\left(1 - \frac{x}{9} + \frac{x^2}{9^2} - \frac{x^3}{9^3} + \ldots\right) = \frac{1}{9} - \frac{x}{9^2} + \frac{x^2}{9^3} - \ldots$$

b) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$; substituting $-x^2$ for x gives
$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \ldots + \frac{(-1)^n x^{2n}}{n!} + \ldots$$

c) $$e^x \cos x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(1 - \frac{x^2}{2} + \ldots\right) = 1 + x + \left(\frac{x^3}{6} - \frac{x^3}{2} + \ldots\right)$$
$$= 1 + x - \frac{x^3}{3} + \ldots;$$ the terms in x^2 cancel.

d) $$\frac{\sin t}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} + \ldots + \frac{(-1)^n t^{2n}}{(2n+1)!} + \ldots$$
$$\int_0^x \frac{\sin t}{t}\,dt = x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot((2n+1)!} + \ldots.$$

e) $$e^{-t^2/2} = 1 - \frac{t^2}{2} + \frac{t^4}{2^2\cdot 2!} - \frac{t^6}{2^3\cdot 3!} + \ldots$$
$$\int_0^x e^{-t^2/2}\,dt = x - \frac{x^3}{3\cdot 2} + \frac{x^5}{5\cdot 2^2\cdot 2!} - \frac{x^7}{7\cdot 2^3\cdot 3!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot 2^n\cdot n!} + \ldots$$

f) $$\frac{1}{x^3-1} = \frac{-1}{1-x^3} = -1 - x^3 - x^6 - \ldots - x^{3n} - \ldots$$

g) $y = \cos^2 x \Rightarrow y' = -2\cos x \sin x = -\sin 2x$; substituting $2x$ into the series for $\sin x$,
$$y' = -2x + \frac{2^3x^3}{3!} - \frac{2^5x^5}{5!} + \ldots;$$ integrating,
$$y = \cos^2 x = -x^2 + \frac{2^3x^4}{4!} - \frac{2^5x^6}{6!} + \ldots + \frac{(-1)^n 2^{2n-1}x^{2n}}{(2n)!} + \ldots + C;$$

Since $y(0) = 1$, we see that $C = 1$, so $\cos^2 x = 1 - x^2 + \frac{x^4}{3} - \ldots$

h) Method 1: $$\frac{\sin x}{1-x} = (\sin x)\left(\frac{1}{1-x}\right) = \left(x - \frac{x^3}{6} + \ldots\right)(1 + x + x^2 + x^3 + \ldots)$$
$$= x + x^2 + \left(x^3 - \frac{x^3}{6} + \ldots\right) = x + x^2 + \frac{5}{6}x^3 + \ldots$$

Method 2: divide $1-x$ into $x - x^3/6 + \ldots$, as done on the left below:

$$\begin{array}{ll} & x + x^2 + 5x^3/6 + \ldots \\ 1-x & x \qquad - x^3/6 \ \ldots \\ & x - x^2 \\ & \quad x^2 - x^3/6 \ + \ldots \\ & \quad x^2 - x^3 \\ & \qquad 5x^3/6 \ + \ldots \end{array} \qquad\qquad \begin{array}{ll} & x + x^3/3 \ + \ldots \\ 1 - x^2/2 & x - x^3/6 \ + \ldots \\ & x - x^3/2 \\ & \qquad x^3/3 \ + \ldots \end{array}$$

i) Method 1: Calculating successive derivatives gives:

$$y = \tan x, \quad y' = \sec^2 x, \quad y'' = 2\sec^2 x \tan x, \quad y^{(3)} = 2(2\sec^2 x \tan x \cdot \tan x + \sec^2 x \cdot \sec^2 x)$$
$$y(0) = 0, \qquad y'(0) = 1, \qquad y''(0) = 0, \qquad y^{(3)}(0) = 2,$$

so the Taylor series starts
$$\tan x = x + \frac{2x^3}{3!} + \ldots = x + \frac{x^3}{3} + \ldots$$

Method 2: $\tan x = \dfrac{\sin x}{\cos x}$; divide the $\cos x$ series into the $\sin x$ series (done on the right above) — this turns out to be easier here than taking derivatives!

7D-3

a) $$\frac{1-\cos x}{x^2} = \frac{1-(1-x^2/2+\ldots)}{x^2} = \frac{x^2/2+\ldots}{x^2} \to \frac{1}{2} \quad \text{as } x \to 0.$$

b) $$\frac{x - \sin x}{x^3} = \frac{x - (x - x^3/6 + \ldots)}{x^3} = \frac{x^3/6 + \ldots}{x^3} \to \frac{1}{6} \quad \text{as } x \to 0$$

c) $$(1+x)^{1/2} = 1 + x/2 - x^2/8 + \ldots \Rightarrow (1+x)^{1/2} - 1 - x/2 = -x^2/8 + \ldots$$
$$\sin x = x - x^3/6 + \ldots \Rightarrow \sin^2 x = x^2 + \ldots$$

Therefore, $$\frac{(1+x)^{1/2} - 1 - x/2}{\sin^2 x} = \frac{-x^2/8 + \ldots}{x^2 + \ldots} \to \frac{-1}{8} \quad \text{as } x \to 0.$$

d) $\cos u - 1 = -u^2/2 + \ldots; \qquad \ln(1+u) - u = -u^2/2 + \ldots;$

Therefore, $$\frac{\cos u - 1}{\ln(1+u) - u} = \frac{-u^2/2 + \ldots}{-u^2/2 + \ldots} \to 1 \quad \text{as } u \to 0.$$